

# Absolute Continuity under Time Shift of Trajectories and Related Stochastic Calculus

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**Abstract** The paper is concerned with a class of two-sided stochastic processes of the form  $X = W + A$ . Here  $W$  is a two-sided Brownian motion with random initial data at time zero and  $A \equiv A(W)$  is a function of  $W$ . Elements of the related stochastic calculus are introduced. In particular, the calculus is adjusted to the case when  $A$  is a jump process. Absolute continuity of  $(X, P_\nu)$  under time shift of trajectories is investigated. For example under various conditions on the initial density with respect to the Lebesgue measure,  $m = d\nu/dx$ , and on  $A$  with  $A_0 = 0$  we verify

$$\frac{P_\nu(dX_{\cdot-t})}{P_\nu(dX_{\cdot})} = \frac{m(X_{-t})}{m(X_0)} \cdot \prod_i |\nabla_{W_0} X_{-t}|_i$$

a.e. where the product is taken over all coordinates. Here  $\sum_i (\nabla_{W_0} X_{-t})_i$  is the divergence of  $X_{-t}$  with respect to the initial position. Crucial for this is the *temporal homogeneity* in the sense that  $X(W_{\cdot+v} + A_v \mathbf{1}) = X_{\cdot+v}(W)$ ,  $v \in \mathbb{R}$ , where  $A_v \mathbf{1}$  is the trajectory taking the constant value  $A_v(W)$ .

By means of such a density, partial integration relative to the generator of the process  $X$  is established. Relative compactness of sequences of such processes is established.

**AMS subject classification (2000)** primary 60G44, 60H07, secondary 60J65, 60J75

**Keywords** Non-linear transformation of measures, anticipative stochastic calculus, Brownian motion, jump processes

## 1 Introduction, Basic Objects, and Main Result

Let us begin with Haar functions over  $[0, 1]$ . Next we do the same on every interval  $[a, a + 1]$  for all integers  $a$  and consider these functions being defined on the entire real axis by extending with zero outside of  $[a, a + 1]$ . We end up with a system of functions  $H_i$ ,  $i \in \mathbb{N}$ , for which we can consider all finite linear combinations

$$\sum_i \xi_i \int_0^t H_i(s) ds, \quad t \in \mathbb{R},$$

where  $\xi_i$ ,  $i \in \mathbb{N}$ , is a sequence of independent  $N(0, 1)$  distributed random variables. The Lévy-Ciesielsky representation of standard Brownian motion tells us now that under proper selection of these finite linear combinations we obtain in the limit a two-sided Brownian

motion  $W$  with  $W_0 = 0$ . The convergence is uniform on every compact subset of  $\mathbb{R}$ , almost surely.

**The initial example.** Denote by  $\varphi$  the density function with respect to the Lebesgue measure of an  $N(0, 1)$  random variable. Let us consider two individual trajectories  $W^1$  and  $W^2$  obtained by the outcomes  $\xi_i = x_i^1$  and, respectively,  $\xi_i = x_i^2$ ,  $i \in \mathbb{N}$ . The weight of  $W^2$  relative to  $W^1$  is  $\prod_{i=1}^{\infty} \frac{\varphi(x_i^2)}{\varphi(x_i^1)}$  provided that this infinite product converges properly. If we assume that  $W_0$  is a random variable independent of  $\xi_i$ ,  $i \in \mathbb{N}$ , with positive density  $m$  then the weight of  $W^2$  relative to  $W^1$  is

$$\frac{m(W_0^2)}{m(W_0^1)} \cdot \prod_{i=1}^{\infty} \frac{\varphi(x_i^2)}{\varphi(x_i^1)}.$$

Now let  $W^2 = W_{-1}^1$ . We obtain by reordering of indices that the weight of  $W^2 = W_{-1}^1$  relative to  $W^1$  is

$$\frac{m(W_0^2)}{m(W_0^1)} \cdot \prod_{i=1}^{\infty} \frac{\varphi(x_i^2)}{\varphi(x_i^1)} = \frac{m(W_{-1}^1)}{m(W_0^1)}.$$

Since the above “reordering” followed by canceling is just a heuristic argument, all what follows from this relation here in the motivating part, is of heuristic nature as well.

The question the paper addresses is what happens if we have a process of type  $X = W + A(W)$  for which we would be interested in such “relative weights” under time shift. We suppose that  $X$  is an injective function of  $W$ . Let us take a look at the very particular case when (i)  $A_0(W) = 0$  for all trajectories  $W$ , (ii) the process  $A(W)$  is independent of  $W_0$ , and (iii) we have  $X_{-1}(W) \equiv W_{-1} + A_{-1}(W) = X(W_{-1} + A_{-1}\mathbb{I})$  where  $A_{-1}\mathbb{I}$  is the function taking the value  $A_{-1}$ , constant in time. Then it is still plausible that the relative weight of  $X_{-1}$  with respect to  $X$  is determined by the weight of  $W_{-1} + A_{-1}\mathbb{I}$  relative to  $W$ , which is by the initial example and (ii)

$$\frac{m(X_{-1})}{m(X_0)} = \frac{m(W_{-1} + A_{-1})}{m(W_0)}.$$

Indeed, given  $W_{-1} + A_{-1}(W)\mathbb{I}$ , according to (iii), we add  $A_{-1}(W) - A_{-1}(W)\mathbb{I} = A(W_{-1} + A_{-1}\mathbb{I})$  to obtain  $X_{-1}(W) = X(W_{-1} + A_{-1}\mathbb{I})$ . On the other hand, given  $W$  we add  $A \equiv A(W)$  to obtain  $X \equiv X(W)$ . We may interpret (iii) as some sort of temporal homogeneity of the in general non-adapted process  $X$ .

Below we are using a more general concept of *temporal homogeneity* of  $X$ . Also, conditions (i) and (ii) will be dropped. However in order to develop a framework that finally results in a Radon-Nikodym density formula for the measure of the trajectories of  $X$  under time shift we need to formulate a set of conditions. On the one hand, these conditions must allow to apply methods from stochastic / Mallianvin calculus and, on the other hand, cover the class of stochastic processes and particle systems we want to discover. Next, we will give verbal answers to the following two questions. What sort of stochastic processes and particle systems are we interested in? What is the major technical challenge we have to overcome to prove the density formula.

**Switching media.** Let us continue the concept of the relative weight of  $X_{-1}$  with respect to  $X$  on an intuitive level. We are now interested in a  $d$ -dimensional process of the form  $X = W + A$  for which (i)  $A_0(W) = 0$  for all trajectories  $W$ , (ii')

$A_s(W) \equiv (A_s^i(W_1, \dots, W_d))_{i=1, \dots, d}$  is for all  $s \in \mathbb{R}$  continuously differentiable with respect to the arguments  $W_1, \dots, W_d$ , and (iii) we have  $X_{-1}(W) \equiv W_{-1} + A_{-1}(W) = X(W_{-1} + A_{-1}\mathbb{1})$ . Let  $\nu$  denote the measure whose density with respect to the  $d$ -dimensional Lebesgue measure is  $m$ , the density of  $W_0$ . The  $d$ -dimensional volume element is in this sense compatibly denoted by  $dW_0$ . Since we do no longer suppose that the process  $A(W)$  is independent of  $W_0$ , it is reasonable that we turn to the Radon-Nikodym derivative  $d\nu(W_{-1} + A_{-1})/d\nu(W_0)$  rather than to look at some ratio of densities. We obtain formally for the relative weight of  $W_{-1} + A_{-1}\mathbb{1}$  with respect to  $W$

$$\begin{aligned} \frac{d\nu(W_{-1} + A_{-1})}{d\nu(W_0)} &= m(W_{-1} + A_{-1}) \frac{d(W_{-1} + A_{-1}(W))}{dW_0} \Big/ m(W_0) \\ &= \frac{m(W_{-1} + A_{-1})}{m(W_0)} \prod_{i=1}^d |1 + \nabla_{W_0} A_{-1}(W)|_i \\ &= \frac{m(X_{-1})}{m(X_0)} \prod_{i=1}^d |\nabla_{W_0} X_{-1}(W)|_i. \end{aligned}$$

Condition (iii) gives rise to consider this term also as the relative weight of  $X_{-1}$  with respect to  $X$ . A motivating example for (iii) is the following. Let  $i_{1,+} \equiv \chi_{\mathbb{R}_{1,+}^d}$  denote the indicator function for the set  $\mathbb{R}_{1,+}^d := \{(x_1, \dots, x_d) : x_1 \geq 0, x_2, \dots, x_d \in \mathbb{R}\}$ , let  $i_{1,+}^\varepsilon$  be some smoothly mollified modification, and let  $a \in \mathbb{R}$ . Let  $X$  be the solution to the Itô SDE

$$X_s = W_s + a \int_0^s i_{1,+}^\varepsilon(X_r) d(W_{1,r}, 0, \dots, 0) =: W_s + A_s, \quad s \in \mathbb{R}.$$

We have (i) by definition and (ii'), because of the above mollification, by a classical theorem of Blagoveshchenskij and Freidlin, cf. [14], Theorem 1.23. For  $s \in \mathbb{R}$  it follows that with  $W^s = W_{+s} + A_s\mathbb{1}$

$$\begin{aligned} X_{\cdot}(W^s) &= W_{\cdot}^s + a \int_0^{\cdot} i_{1,+}^\varepsilon(X_r(W^s)) d(W_{1,r}^s, 0, \dots, 0) \\ &= W_{\cdot}^s + a \cdot \int_0^{\cdot} i_{1,+}^\varepsilon(X_r(W^s)) d(W_{1,s+r}, 0, \dots, 0) \end{aligned}$$

and

$$\begin{aligned} X_{+s}(W) &= W_{+s} + a \int_0^{+s} i_{1,+}^\varepsilon(X_r(W)) d(W_{1,r}, 0, \dots, 0) \\ &= W_{+s} + A_s + a \int_s^{+s} i_{1,+}^\varepsilon(X_r(W)) d(W_{1,r}, 0, \dots, 0) \\ &= W_{\cdot}^s + a \int_0^{\cdot} i_{1,+}^\varepsilon(X_{r+s}(W)) d(W_{1,r+s}, 0, \dots, 0) \end{aligned}$$

which implies  $X_{\cdot}(W^s) = X_{+s}(W)$ ,  $s \in \mathbb{R}$ . In other words, we have (iii). The choice  $a \in (-1, 0)$  (respectively  $a \in (0, \infty)$ ) lets  $X$  move slower (respectively faster) than  $W$  on  $\{(x_1, \dots, x_d) : x_1 \geq \delta, x_2, \dots, x_d \in \mathbb{R}\}$ . Here  $\delta > 0$  is some small parameter depending on the above mollification.

**The particle collision type jumps.** The stochastic processes and particle systems we are focusing on may involve jumps. Definition 1.7 formulates the mathematical conditions on the jumps. In this paragraph we give a physical motivation for Definition 1.7.

Let us consider a system of  $n$  marked particles  $\{X_{o1}, \dots, X_{on}\}$  in some  $d_o$ -dimensional domain  $D_o$ . If we model the particles just as points in  $D_o$  and consequently the collection of all  $n$  particles as a point in  $\mathbb{R}^{n \cdot d_o}$  then the set of all potential collisions is the set  $Z_o$  defined as follows. For  $z_1, \dots, z_n \in D_o$ ,  $z = (z_1, \dots, z_n)$ , and  $i \in \{1, \dots, n\}$  introduce  $z^{ij} := (z_1, \dots, z_{i-1}, z_j, z_{i+1}, \dots, z_n)$ ,  $i, j \in \{1, \dots, n\}$ ,  $i < j$ . Now, let

$$Z_o(i, j) := \{z^{ij} : z \in D_o^n\}, \quad i < j, \quad Z_o := \bigcup_{i < j} Z_o(i, j).$$

If we look at particles as small  $d_o$ -dimensional objects then a counterpart is  $Z_o^\varepsilon(i, j) := \{z \in \mathbb{R}^{n \cdot d_o} : |z - z'| \leq \varepsilon, z' \in Z_o(i, j)\}$ ,  $i < j$ , and  $Z_o^\varepsilon := \bigcup_{i < j} Z_o^\varepsilon(i, j)$ . In this case however, we choose from the *physical particle* some canonical inner point, the *mathematical particle*; for example the midpoint if the physical particle is a small  $d_o$ -dimensional ball.

Now we extend the notion of a particle to an object having a  $d_o$ -dimensional geometric location and a  $d_v$ -dimensional velocity, a vector belonging to the difference  $D_v$  of the  $d_v$ -dimensional open ball with center 0 and some radius  $v_{\max} < \infty$  and the closed ball with center 0 and some radius  $v_{\min} < v_{\max}$ . Usually we have  $d_o = d_v$ . The system of mathematical particles we are now interested in is  $\{X_1, \dots, X_n\}$  where  $X_i := (X_{oi}, X_{vi})$  and  $X_{vi}$  is the velocity of  $X_{oi}$ . We set  $d := d_o + d_v$  and  $D := D_o \times D_v$ . In this sense, the set of all potential collisions of particle  $i$  with particle  $j$ ,  $i < j$ , is now  $Z(i, j) := Z_o(i, j) \times D_v$  or, if the physical location of the particle is a small  $d_o$ -dimensional set,  $Z^\varepsilon(i, j) := Z_o^\varepsilon(i, j) \times D_v$ . Respectively, we set  $Z := \bigcup_{i < j} Z(i, j)$  and  $Z^\varepsilon := \bigcup_{i < j} Z^\varepsilon(i, j)$ .

Let  $\xi(s)$ ,  $s \in \mathbb{R}$ , be an  $\mathbb{R}^{n \cdot d}$ -valued trajectory describing the path of a system of  $n$  mathematical particles. Furthermore, for  $x \in \mathbb{R}^{n \cdot d}$ , write  $x\mathbb{I} \equiv x\mathbb{I}(s)$  for the trajectory which takes the value  $x$  for all  $s \in \mathbb{R}$ . By the concept physical and mathematical particles,  $\xi_s$ ,  $s \in \mathbb{R}$ , never reaches the boundary  $\partial D^n$ . Near the boundary  $\partial D^n$  we assume that the trajectory  $\xi$  models some reflection / adhesion / diffusion mechanism of the particle(s) that is (are) closest to  $\partial D$ .

For small  $z \in \mathbb{R}^{n \cdot d}$ , let us also consider the *parallel trajectory*  $\xi_z(s)$ ,  $s \in \mathbb{R}$ , whenever it meaningfully exists. That is  $\xi_z = \xi + z\mathbb{I}$ , as long as  $\xi + z\mathbb{I}$  has not got *close* to  $\partial D^n$  and is again  $\xi + z\mathbb{I}$  as soon as it is not close to  $\partial D^n$  anymore. The latter could also mean “as soon as two individual mathematical particles with the same location perform a collision rather than following some intrinsic mechanism near the boundary”.

Until the end of this paragraph we would like to keep the interpretation that for  $z = (z_1, \dots, z_n)$  and  $z_i = (z_{oi}, z_{vi})$ , the component  $z_{oi} \in \mathbb{R}^{d_o}$  stands for location and the component  $z_{vi} \in \mathbb{R}^{d_v}$  stands for velocity of  $z_i$ . We make the following observation. It is reasonable to assume that for the trajectory  $\xi$  and the parallel trajectory  $\xi + z\mathbb{I}$  we have  $z_{vi} = 0$ ,  $i \in \{1, \dots, n\}$ . However we would like to have  $\xi_x$  be well-defined for  $x$  in some neighborhood of  $x = 0$  and set therefore artificially  $\xi_x := \xi_{(x_o, 0)}$  where  $x_o := (x_{1o}, \dots, x_{no}) \in \mathbb{R}^{n \cdot d_o}$  is the vector of the locations of the  $x_i \in \mathbb{R}^d$ ,  $i \in \{1, \dots, n\}$ .

Let  $\tau(x)$  denote the first time after 0 for  $\xi_x$  two particles collide, that is, the first time after 0 the trajectory  $\xi_x$  hits  $Z^\varepsilon$ . The function  $x \rightarrow \tau(x)$  is then a piecewise infinitely boundedly differentiable function in some neighborhood of  $x = 0$ . The term *piecewise* refers to the different possibilities for the first collision to be caused by the particles  $i$  and  $j$ ,  $i < j$ . On the pieces,  $\nabla_x \tau(x) \in \mathbb{R}^{n \cdot d}$  is by the above convention well-defined and we have  $\langle \nabla_x \tau(0), e \rangle_{\mathbb{R}^{n \cdot d}} = 0$  for any vector  $e = ((e_{1o}, e_{1v}), \dots, (e_{no}, e_{nv}))$  with  $e_{1o} = \dots = e_{no} = 0$  where  $e_{io}$ ,  $i \in \{1, \dots, n\}$ , are the locations of the  $e_i \in \mathbb{R}^d$ ,  $i \in \{1, \dots, n\}$ .

By the trajectory  $\xi$  we follow  $n$  particles. For those we may assume that immediately after the first collision of two, the vector of the velocities has changed by a jump while the vector of the locations is unchanged. Thus, for  $\Delta\xi_\tau := \xi_{\tau+} - \xi_{\tau-}$  we get

$$\langle \nabla_x \tau(0), \Delta\xi_\tau \rangle_{\mathbb{R}^{n \cdot d}} = 0.$$

This is motivation for Definition 1.7 (jv). Condition (jv) of Definition 1.7 is at the same time a technical hypothesis.

**The process  $Y$ .** Instead of (i) and (iii) formulated here in the introductory section, for the actual analysis in the paper we will use condition of (3) of Subsection 1.2. This condition does no longer require  $A_0 = 0$ . It includes a process  $Y$  with  $A_0 = Y_0$  such that with  $W^s := W_{+s} + (A_s(W) - Y_s(W))\mathbb{1}$  we have the relation  $X.(W^s) = X_{+s}(W)$ ,  $s \in \mathbb{R}$ . Below we give a reason why (i) and (iii) is not enough.

Let  $X$  be a  $n \cdot n$ -dimensional stochastic particle process of the form  $X = W + A$  where  $A \equiv A(W)$  is a right-continuous pure jump process. That is,  $A$  is constant in time until it jumps and thereafter constant again until it jumps again. The jump times and jumps are measurable functions of  $W$ . Assume that the jumps are organized as in the previous paragraph. This means in particular that whenever  $X$  hits  $Z^\varepsilon$  at some jump time it performs a jump within  $Z^\varepsilon$  and keeps moving afterwards as it was a  $n \cdot n$ -dimensional Brownian motion. For proper modeling we must here assume that jump times do not accumulate. However the following scenario is possible. The trajectory  $W$  passes through  $Z^\varepsilon$  in a subset of  $Z^\varepsilon$  no jump is allowed to reach. This happens at some negative time and no further jump occurs until time zero. Consequently,  $A_0 = 0$  which is equivalent to  $X_s = W_s$  in some non-positive time interval  $s \in (\sigma, 0]$ , is not possible. In order to construct an injection  $W \rightarrow X$  we must allow  $A_0 \neq 0$ . In that case  $X.(W^s) = X_{+s}(W)$ ,  $s \in \mathbb{R}$ , for  $W^s := W_{+s} + A_s\mathbb{1}$  may fail.

**The integral  $\int_{s \in I} F(s, W) d\dot{W}_s$**  for some interval  $I$ . This type of integral appears unavoidably in the beginning of the proof of our main result, namely in the second last line of (3.110). We handle this integral by projecting  $W$  via the Lévy-Ciesielsky representation to a piecewise linear path. The critical operations on  $\int_{s \in I} F(s, W) d\dot{W}_s$  will be carried out under the projection. Then we apply several extension techniques but mainly the approximation theorem Theorem 2.1 by Gihman and Skorohod, cf. [6]. The calculus on the projections is developed in Subsections 2.1, 2.3, 3.1, and 3.2.

**Organization of the paper.** After this motivating part, we collect all symbols and definitions we frequently need in the paper. The concept of two-sided Brownian motion is taken from Imkeller [8]. Symbols and definitions just for momentary use will be introduced when they are needed. In Subsection 1.2 we formulate a list of conditions we need in order to state the main result, the change of measure formula under time shift of trajectories. The main results are presented in Subsection 1.3. Immediate corollaries are formulated and proved in this subsection.

Section 2 contains the stochastic calculus on the above mentioned projections of trajectories. The results of Subsections 2.1 and 2.3 are important for the proof of the change of measure formula. Subsection 2.2 formulates and proves a related stochastic calculus where the time shift is replaced by a certain class of trajectory valued flows. Here we refer to Mayer-Wolf and Zakai [11] and Smolyanov and Weizsäcker [18] for related ideas.

If the trajectory flow is a translation along the time axis, we shift piecewise linear trajectories, obtained by projections on the Lévy-Ciesielsky representation. This operation

requires particular attention to objects like gradient, stochastic integral, and generator of the flow. The related assertions are proved in Subsections 3.1 and 3.2.

The proof of the change of measure formula involves two approximations. The first one corresponds to the projection via the Lévy-Ciesielsky representation. The second one is the approximation of the trajectory valued flow  $s \rightarrow W^s = W_{\cdot+s} + (A_s - Y_s)\mathbb{I}$  if  $A - Y$  has jumps by a sequence of continuous flows. The change of measure formula is proved in Subsection 3.4. We would like to refer to Bogachev and Mayer-Wolf [2], Buckdahn [3], Kulik and Pilipenko [10].

Sections 4 and 5 contain two applications. In Section 4 we prove integration by parts formulas for operators which are for, in general, non-Markov processes counterparts to generators of Markov processes. In Section 5 we arrive at the application the paper was initiated by, relative compactness of a class of abstract particle systems. This application may explain the setup, conditions, and notations in the paper. It has been motivated by concrete physical modeling as in Caprino, Pulvirenti, and Wagner [4] as well as a more abstract mathematical and physical treatment as in Graham and Méléard [7].

Before starting this preliminary part of the paper, let us emphasize that most of the terms related to the Malliavin calculus and stochastic integration are explained in Section 6. Section 6 is independent and should at least be browsed prior to reading this introduction.

## 1.1 Analytical setting

**The symbol  $F$ .** Let  $n, d \in \mathbb{N}$ . In order to simplify the notation, we will use the letter  $F$  in order to denote both, the space of all real  $n \cdot d$ -dimensional vectors  $(z_1, \dots, z_n)$  with  $z_1, \dots, z_n \in \mathbb{R}^d$  and the space of all real  $n \cdot d \times n \cdot d$ -matrices. Correspondingly, we will use the symbol  $\langle \cdot, \cdot \rangle_F$  in order to denote the square of a vector norm or of a matrix norm on  $F$  which we assume to be compatible but do not specify. We also assume  $\langle \cdot, \cdot \rangle_F^{1/2}$  to be submultiplicative if  $F = \mathbb{R}^{n \cdot d} \otimes \mathbb{R}^{n \cdot d}$ .

Let  $\lambda_F$  denote the Lebesgue measure on  $F$  and let  $(e_j)_{j=1, \dots, n \cdot d}$  be a standard basis in  $\mathbb{R}^{n \cdot d}$ . Moreover, let  $\mathbf{e} = \sum_{i=1}^{n \cdot d} e_j$  if  $F = \mathbb{R}^{n \cdot d}$ . Let  $\mathbf{e}$  be the  $n \cdot d \times n \cdot d$ -dimensional unit matrix if  $F = \mathbb{R}^{n \cdot d} \otimes \mathbb{R}^{n \cdot d}$ .

Furthermore, depending on the format of the entries, let us define the product  $\langle \cdot, \cdot \rangle_{F \rightarrow F}$ . If both entries belong to  $F = \mathbb{R}^{n \cdot d}$ , we set

$$\langle (x_1, \dots, x_{n \cdot d}), (y_1, \dots, y_{n \cdot d}) \rangle_{F \rightarrow F} := \sum_{i=1}^{n \cdot d} x_i y_i \cdot e_i, \quad x_1, \dots, x_{n \cdot d}, y_1, \dots, y_{n \cdot d} \in \mathbb{R}.$$

If at least one of the entries belongs to  $F = \mathbb{R}^{n \cdot d} \otimes \mathbb{R}^{n \cdot d}$  and not otherwise stated, let  $\langle \cdot, \cdot \rangle_{F \rightarrow F}$  denote the usual matrix-vector, vector-matrix, or matrix-matrix multiplication.

Set  $\mathbb{I}(s) := \mathbf{e}$ ,  $s \in \mathbb{R}$ , and for  $B \in \mathcal{B}(\mathbb{R})$  set  $\mathbb{I}_B(s) = \mathbf{e}$  if  $s \in B$  and  $\mathbb{I}_B(s) = 0$  if  $s \notin B$ . If  $x \in F$  then for simplicity, we will use the short notation

$$x\mathbb{I} \equiv \langle x, \mathbb{I}(\cdot) \rangle_{F \rightarrow F}.$$

It will be always clear from the context which of the spaces,  $\mathbb{R}^{n \cdot d}$  or  $\mathbb{R}^{n \cdot d} \otimes \mathbb{R}^{n \cdot d}$ , we talk about when we use the symbol  $F$  and which form of multiplication  $\langle \cdot, \cdot \rangle_{F \rightarrow F}$  we use.

**Function spaces and orthogonal projections.** Among others, which will be introduced upon usage, we will work with the spaces  $L^2(\mathbb{R}; F)$  and  $L^2_{\text{loc}}(\mathbb{R}; F)$  of all quadratically

integrable and, respectively, local quadratically integrable  $F$ -valued functions on  $\mathbb{R}$ . For simplicity, the latter space will be abbreviated just by  $L^2$  if no ambiguity is possible. For  $\int \langle a(s), b(s) \rangle_{F \rightarrow F} ds$ ,  $a, b \in L^2(\mathbb{R}; F)$ , we shall use the symbol  $\langle a, b \rangle_{L^2 \rightarrow F}$ . Moreover, for an integral of type  $\int_{-\infty}^{\infty} \langle a, db \rangle_F$  or  $\int_{-\infty}^{\infty} \langle a, db \rangle_{F \rightarrow F}$  we shall use the short notation  $\langle a, db \rangle_{L^2}$  or  $\langle a, db \rangle_{L^2 \rightarrow F}$ , again if no ambiguity is possible.

In addition, let  $C^k(\mathbb{R}; F)$  be the space of all  $k$  times differentiable  $F$ -valued functions on  $\mathbb{R}$ . When adding a subscript zero,  $C_0^k(\mathbb{R}; F)$ , we restrict ourself to compactly supported elements. Furthermore, let  $C_a(\mathbb{R}; F)$  denote the space of all absolutely continuous  $F$ -valued functions such that, for  $f \in C_a(\mathbb{R}; F)$ , its Radon-Nikodym derivative admits a cadlag version. In fact, the cadlag version is at the same time the right derivative of  $f \in C_a(\mathbb{R}; F)$  and will be denoted by  $f'$ . When restricting to functions defined just on some subset  $\mathcal{S} \subset \mathbb{R}$ , we simply replace  $\mathbb{R}$  in these symbols by  $\mathcal{S}$ .

Let  $B_{b,\text{loc}}(\mathbb{R}; F)$  denote the space of all Lebesgue measurable  $F$ -valued functions on  $\mathbb{R}$  possessing a version that is bounded on every finite subinterval of  $\mathbb{R}$ .

Finally, let  $C^k(E)$  be the space of all  $k$  times continuously differentiable real functions on  $E \subseteq F$ . When adding here a subscript  $b$ ,  $C_b^k(E)$ , we just count in the bounded functions of  $C^k(E)$  with bounded derivatives up to order  $k$ .

Let  $t > 0$ . Moreover, let

$$D_1(s) = t^{-\frac{1}{2}}, \quad s \in [0, t), \quad D_1(s) = 0, \quad s \in \mathbb{R} \setminus [0, t),$$

$$D_{2^m+k}(s) = \begin{cases} t^{-\frac{1}{2}} \cdot 2^{\frac{m}{2}} & \text{if } s \in [\frac{k-1}{2^m} \cdot t, \frac{2k-1}{2^{m+1}} \cdot t) \\ -t^{-\frac{1}{2}} \cdot 2^{\frac{m}{2}} & \text{if } s \in [\frac{2k-1}{2^{m+1}} \cdot t, \frac{k}{2^m} \cdot t) \\ 0 & \text{if } s \in \mathbb{R} \setminus [\frac{k-1}{2^m} \cdot t, \frac{k}{2^m} \cdot t) \end{cases},$$

$$k \in \{1, \dots, 2^m\}, \quad m \in \mathbb{Z}_+.$$

Furthermore, let  $f_j$ ,  $j \in \{1, \dots, n \cdot d\}$ , be a standard basis in  $\mathbb{R}^{n \cdot d}$ , and define

$$E_{n \cdot d \cdot (q-1)+j} := D_q \cdot f_j, \quad q \in \mathbb{N}, \quad j \in \{1, \dots, n \cdot d\},$$

as well as

$$E_k^{(w)}(s) := E_k(s + (w-1)t), \quad s \in \mathbb{R}, \quad k \in \mathbb{N}, \quad w \in \mathbb{Z},$$

and rename the elements of  $\{E_k^{(w)} : k \in \mathbb{N}, w \in \mathbb{Z}\}$  as  $\{H_1, H_2, \dots\}$ .

Let  $I(m, r)$  be the set of all indices such that  $\{H_i : i \in I(m, r)\}$  is the set of all  $H_j$  with  $H_j(s) = 0$ ,  $s \notin [-rt, (r-1)t]$ , and  $\text{supp } H_j$  is a closed interval of the form  $[\frac{k-1}{2^{m'}} \cdot t, \frac{k}{2^{m'}} \cdot t]$ ,  $0 \leq m' \leq m-1$ ,  $m \in \mathbb{Z}_+$ ,  $r \in \mathbb{N}$ . Furthermore, set  $J(m) := \bigcup_{r \in \mathbb{N}} I(m, r)$ ,  $m \in \mathbb{Z}_+$  and  $I(r) := \bigcup_{m \in \mathbb{Z}_+} I(m, r)$ ,  $r \in \mathbb{N}$ .

Let  $p_{m,r}$ , resp.  $p_m$ , denote the orthogonal projection from  $L_{\text{loc}}^2(\mathbb{R}; F)$  to the linear subspace spanned by  $\{H_i : i \in I(m, r)\}$ , resp.  $\{H_i : i \in J(m)\}$ .

Let  $\mathcal{R} \equiv \mathcal{R}(r) := [-rt, (r-1)t]$ . For calculations under the measure  $Q_\nu^{(m,r)}$  we will use the set

$$H \equiv H^{(r)} := \{(g, y) : g \in L^2(\mathcal{R}; F), y \in F\}.$$

Let  $l_{\mathcal{R}}$  denote that real measure on  $\mathbb{R}$  which is the Lebesgue measure on  $\mathcal{R}$  and zero on  $\mathbb{R} \setminus \mathcal{R}$ . In addition, identify  $H$  with the space  $\{(g, y) : g \in L^2(\mathbb{R}, l_{\mathcal{R}}; F), y \in F\}$ . The inner product in  $H$  is  $\langle h_1, h_2 \rangle_H = \langle f_1, f_2 \rangle_{L^2} + \langle x_1, x_2 \rangle_F$  but let us also use the notation

$$\langle h_1, h_2 \rangle_{H \rightarrow F} := \langle f_1, f_2 \rangle_{L^2 \rightarrow F} + \langle x_1, x_2 \rangle_{F \rightarrow F}, \quad h_i = (f_i, x_i) \in H, \quad i \in \{1, 2\}.$$

Again, we embed  $H$  into  $C(\mathbb{R}; F)$  by  $j(f, x) := (\int_0^\cdot f(s) ds, x) \equiv x\mathbb{I} + \int_0^\cdot f(s) ds$ . The latter gives also rise to set

$$j^{-1}g := (g', g(0)), \quad g \in C_a(\mathbb{R}; F) \text{ with } g' \in L^2(\mathbb{R}, l_{\mathcal{R}}; F).$$

Finally, let  $H^{(m,r)}$  resp.  $H^{(m)}$  be the linear space spanned by  $\{(H_i, x) : i \in I(m, r), x \in F\}$  resp.  $\{(H_i, x) : i \in J(m), x \in F\}$ .

## 1.2 Probabilistic setting

**Two-sided Brownian motion with random initial data.** Let  $\nu$  be a probability distribution on  $(F, \mathcal{B}(F))$ , here  $F = \mathbb{R}^{n-d}$ . Let the probability space  $(\Omega, \mathcal{F}, Q_\nu)$  be given by the following.

- (i)  $\Omega = C(\mathbb{R}; F)$ , the space of all continuous functions  $W$  from  $\mathbb{R}$  to  $F$ . Identify  $C(\mathbb{R}; F) \equiv \{\omega \in C(\mathbb{R}; F) : \omega(0) = 0\} \times F$ , here also  $F = \mathbb{R}^{n-d}$ .
- (ii)  $\mathcal{F}$ , the  $\sigma$ -algebra of Borel sets with respect to uniform convergence on compact subsets of  $\mathbb{R}$ .
- (iii)  $Q_\nu$ , the probability measure on  $(\Omega, \mathcal{F})$  for which, when given  $W_0$ , both  $(W_s - W_0)_{s \geq 0}$  as well as  $(W_{-s} - W_0)_{s \geq 0}$  are independent standard Brownian motions with state space  $(F, \mathcal{B}(F))$ . Assume  $W_0$  to be independent of  $(W_s - W_0)_{s \in \mathbb{R}}$  and distributed according to  $\nu$ .

In addition, we will assume that the natural filtration  $\{\mathcal{F}_u^v = \sigma(W_\alpha - W_\beta : u \leq \alpha, \beta \leq v) \times \sigma(W_0) : -\infty < u < v < \infty\}$  is completed by the  $Q_\nu$ -completion of  $\mathcal{F}$ .

For the measure  $\nu$  we shall assume the following throughout the paper.

- (i)  $\nu$  admits a density  $m$  with respect to  $\lambda_F$  and  $m$  is symmetric with respect to the  $n$   $d$ -dimensional components of  $F$ .
- (ii)  $m$  is supported by a set  $\overline{D^n}$  where  $D$  is a bounded  $d$ -dimensional domain such that

$$0 < m \in C^1(D^n) \quad \text{and} \quad \lim_{D^n \ni x \rightarrow \partial D^n} m(x) = 0.$$

- (iii) Furthermore, let us assume that there exist  $Q \in (1, \infty)$  such that, for all  $y \in F$ ,

$$\frac{d}{d\lambda} \bigg|_{\lambda=0} \frac{m(\cdot + \lambda y)}{m} = \left\langle \frac{\nabla m}{m}, y \right\rangle_F \quad \text{exists in } L^Q(F, \nu).$$

**Measure, gradient, and stochastic integral under projections.** Recalling the projections on subspaces spanned by sets of Haar functions, for  $(f, x) \in L^2 \times F$ , we set

$q_{m,r}(f, x) := (p_{m,r}f, x)$  and  $q_m(f, x) := (p_m f, x)$ . For  $\Omega \ni W = y + \sum_{i=1}^{\infty} x_i \cdot \int_0^{\cdot} H_i(u) du$  such that the sum converges uniformly on finite intervals we will use the notation

$$q_{m,r}j^{-1}(W) := \left( \sum_{i \in I(m,r)} x_i \cdot H_i, y \right) \quad \text{and} \quad q_m j^{-1}(W) := \left( \sum_{i \in J(m)} x_i \cdot H_i, y \right)$$

which is compatible with the operator  $j$  defined in the end of Subsection 1.1 and the just defined projections  $q_{m,r}$  and  $q_m$ . The sum  $\sum_{i \in J(m)} x_i \cdot H_i$  is to be understood in the sense  $L_{\text{loc}}^2(\mathbb{R}; F)$ . We are now able to introduce the projections

$$\pi_{m,r}(W) := j q_{m,r} j^{-1}(W) \quad \text{and} \quad \pi_m(W) := j q_m j^{-1}(W), \quad W \in \Omega.$$

Furthermore, we introduce the measure  $Q_{\nu}^{(m,r)}$  on  $(\Omega, \mathcal{B}(\Omega))$  by

$$Q_{\nu}^{(m,r)}(\Omega \setminus \{\pi_{m,r}(W) : W \in \Omega\}) = 0$$

and, on  $\{\pi_{m,r}(W) : W \in \Omega\}$ , by

$$Q_{\nu}^{(m,r)} = Q_{\nu} \circ \pi_{m,r}^{-1}, \quad m \in \mathbb{Z}_+, \quad r \in \mathbb{N}.$$

Similarly, we define the measure  $Q_{\nu}^{(m)}$  on  $(\Omega, \mathcal{B}(\Omega))$  by

$$Q_{\nu}^{(m)}(\Omega \setminus \{\pi_m(W) : W \in \Omega\}) = 0$$

and, on  $\{\pi_m(W) : W \in \Omega\}$ , by

$$Q_{\nu}^{(m)} = Q_{\nu} \circ \pi_m^{-1}, \quad m \in \mathbb{Z}_+.$$

Likewise, but without indicating this in the notation, let us also define the measure  $Q_{\nu}^{(m,r)}$  on Borel subsets  $\Omega' \in \mathcal{B}(\Omega)$  with  $\{\pi_{m,r}(W) : W \in \Omega\} \subseteq \Omega'$  and the measure  $Q_{\nu}^{(m)}$  on  $\Omega'$  whenever  $\{\pi_m(W) : W \in \Omega\} \subseteq \Omega'$ .

**Definition 1.1** (a) Let  $f \in C_b^1(\mathbb{R}^{\#I(m,r)+n \cdot d})$ ,  $\langle e, W \rangle$  be the vector of all  $\langle e_j, W_0 \rangle_F$ ,  $j \in \{1, \dots, n \cdot d\}$ , and  $\langle H, W \rangle$  be the vector of all  $\langle H_i, dW \rangle_{L^2}$ ,  $i \in I(m, r)$ ,  $W \in C(\mathbb{R}; F)$ . Let  $\varphi(W) := f(\langle H, W \rangle, \langle e, W \rangle)$ . Define the *gradient of a cylindrical function* with respect to the measure  $Q_{\nu}^{(m,r)}$  by

$$\begin{aligned} \mathbb{D}\varphi(W)(j(g, x)) &:= \sum_{i \in I(m,r)} f_i(\langle H, W \rangle, \langle e, W \rangle) \cdot \langle H_i, g \rangle_{L^2} \\ &\quad + \sum_{j=1}^{n \cdot d} f_j(\langle H, W \rangle, \langle e, W \rangle) \cdot \langle e_j, x \rangle_F \end{aligned}$$

where  $f_i$  denotes the first order derivative of  $f$  relative to  $i \in I(m, r)$  and  $f_j$  is the first order derivative of  $f$  relative to  $j \in \{1, \dots, n \cdot d\}$ .

We remark that Definition 1.1 (a) is in compliance with (6.4) and (6.5) by choosing  $l_i := (-dH_i, 0)$ ,  $i \in I(m, r)$ , and  $l_j := (0, e_j)$ ,  $j \in \{1, \dots, n \cdot d\}$ . Let  $\mathcal{C} \equiv \mathcal{C}(\Omega)$  denote the set of all cylindrical functions of the above form. Let  $Q$  be the number appearing in the

definition of the density  $m$  and  $1/p + 1/Q \leq 1$ .  $\mathcal{C}$  is dense in  $L^p(\Omega, Q_\nu^{(m,r)})$ . As an adaption of Proposition 6.3, the operator  $(D, \mathcal{C}) := (j^* \circ \mathbb{D}, \mathcal{C})$  is closable on  $L^p(\Omega, Q_\nu^{(m,r)}; H)$ .

**Definition 1.1 continued.** (b) Let  $(D, D_{p,1}) \equiv (D, D_{p,1}(Q_\nu^{(m,r)}))$  denote the closure of  $(D, \mathcal{C}) := (j^* \circ \mathbb{D}, \mathcal{C})$  on  $L^p(\Omega, Q_\nu^{(m,r)}; H)$ .

(c) If no ambiguity is possible we will also use the symbol  $(D, D_{p,1})$  to denote the (vectors of) gradients of functions of type  $\Omega \rightarrow F$ .

(d) Let  $\varphi \in D_{p,1}$  be of type  $\Omega \rightarrow \mathbb{R}$ . If  $\varphi$  is for  $W \in \Omega$  of the form  $\varphi(W - W_0 \mathbb{1}, W_0)$  and the  $n \cdot d$ -dimensional vector of the directional derivatives of  $\varphi$  in the directions of the components of  $W_0$  exists in the sense of differentiation in  $L^p(\Omega, Q_\nu^{(m,r)}; F)$  then let it be denoted by  $\nabla_{W_0} \varphi \equiv \nabla_{W_0} \varphi(W)$ . When emphasizing that  $D\varphi(W) \in H$  for  $Q_\nu^{(m,r)}$ -a.e.  $W \in \Omega$  or stressing the particular form of the gradient we will use the notation

$$D\varphi(W) \equiv \left( (D\varphi)_1(W), (D\varphi)_2(W) \right)$$

or

$$D\varphi(W) \equiv \left( (D\varphi)_1(W), \nabla_{W_0} \varphi(W) \right).$$

(e) Let  $f \in \bigcup_{k=1}^{\infty} C_b^1(\mathbb{R}^{k+n \cdot d})$  and  $W \in C(\mathbb{R}; F)$ . Let  $\langle e, W \rangle$  be the vector of all  $\langle e_j, W_0 \rangle_F$ ,  $j \in \{1, \dots, n \cdot d\}$ . Furthermore, let  $k \in \mathbb{N}$ ,  $i_1, \dots, i_k \in J(m)$ , and  $\langle H, W \rangle$  be the vector of all  $\langle H_{i_l}, dW \rangle_{L^2}$ ,  $l \in \{1, \dots, k\}$ . Let  $\varphi(W) := f(\langle H, W \rangle, \langle e, W \rangle)$ . Define here the *gradient of a cylindrical function* with respect to the measure  $Q_\nu^{(m)}$  by

$$\begin{aligned} \mathbb{D}\varphi(W)(j(g, x)) &:= \sum_{l \in \{1, \dots, k\}} f_l(\langle H, W \rangle, \langle e, W \rangle) \cdot \langle H_{i_l}, g \rangle_{L^2} \\ &+ \sum_{j=1}^{n \cdot d} f_j(\langle H, W \rangle, \langle e, W \rangle) \cdot \langle e_j, x \rangle_F \end{aligned}$$

where  $f_l$  denotes the first order derivative of  $f$  relative to the index  $i_l$ ,  $l \in \{1, \dots, k\}$ , and  $f_j$  is the first order derivative of  $f$  relative to  $j \in \{1, \dots, n \cdot d\}$ . Parts (b)-(d) can now be modified to the definition of  $(D, D_{p,1}(Q_\nu^{(m)}))$  by replacing  $I(m, r)$  with  $J(m)$ ,  $Q_\nu^{(m,r)}$  with  $Q_\nu^{(m)}$ , and  $H$  with  $\{(g, y) : g \in L^2(\mathbb{R}; F), y \in F\}$  (with inner product  $\langle h_1, h_2 \rangle = \langle f_1, f_2 \rangle_{L^2} + \langle x_1, x_2 \rangle_F$ ).

Referring to part (d), it is important to note that, for  $\varphi \in D_{p,1}$ ,  $D\varphi$  is  $Q_\nu^{(m,r)}$ -a.e a linear combination of  $\langle H_{i_l}, \cdot \rangle_{L^2}$ ,  $i \in I(m, r)$ , and  $\langle e_j, \cdot \rangle_F$ ,  $j \in \{1, \dots, n \cdot d\}$ , with random coefficients. For this, see also Lemma 2.2 (a) below.

**Definition 1.1 continued.** (f) Let  $1/q' + 1/Q \leq 1$ ,  $1/p' + 1/q' = 1$ , and  $\xi \in L^{p'}(\Omega, Q_\nu^{(m,r)}; H)$ . We say that  $\xi \in \text{Dom}_{p'}(\delta)$  if there exists  $c_{p'}(\xi) > 0$  such that

$$\int \langle D\varphi, \xi \rangle_H dQ_\nu^{(m,r)} \leq c_{p'}(\xi) \cdot \|\varphi\|_{L^{q'}(\Omega, Q_\nu^{(m,r)})}, \quad \varphi \in D_{q',1}(Q_\nu^{(m,r)}).$$

In this case, we define the stochastic integral  $\delta(\xi) \equiv \delta^{(m,r)}(\xi)$  by

$$\int \delta(\xi) \cdot \varphi dQ_\nu^{(m,r)} = \int \langle \xi, D\varphi \rangle_H dQ_\nu^{(m,r)}, \quad \varphi \in D_{q',1}(Q_\nu^{(m,r)}).$$

The representations of Theorem 6.7 apply. Furthermore,  $\delta(\xi) \in L^{p'}(\Omega, Q_\nu^{(m,r)})$  by the Hahn-Banach theorem. Furthermore, in order to motivate this definition, we note that  $D_{q',1}(Q_\nu^{(m,r)})$  is defined for  $1/q' + 1/Q \leq 1$ , recall also Proposition 6.3.

**Gradients with respect to an individual trajectory** Let  $\xi \equiv \xi(W)$  be an  $\hat{F}$ -valued random variable with  $\hat{F}$  being a metric space.

**Definition 1.2** (a) For fixed  $W \in \Omega$  and  $x \in F$ , let

$$\nabla_x \xi(W + x\mathbb{I}) := \sum_{j=1}^{n \cdot d} \frac{\partial}{\partial x_j} \xi(W + x\mathbb{I}) \cdot (0, e_j)$$

whenever the partial derivatives  $\frac{\partial}{\partial x_j} \xi(W + x\mathbb{I}) = \lim_{h \rightarrow 0} \frac{1}{h} (\xi(W + x\mathbb{I} + h e_j \mathbb{I}) - \xi(W + x\mathbb{I}))$  exist in  $\hat{F}$  for all  $j \in \{1, \dots, n \cdot d\}$ .

(b) Set

$$\nabla_{W_0} \xi(W) := \nabla_x|_{x=W_0} \xi(W - W_0\mathbb{I} + x\mathbb{I}).$$

(c) For  $W \in \Omega$  and  $\kappa_i := \int_0^\cdot H_i(v) dv$  let

$$\nabla_G \xi(W) := \sum_{i \in I(r)} \frac{\partial}{\partial \kappa_i} \xi(W) \cdot (H_i, 0)$$

whenever the partial derivatives  $\frac{\partial}{\partial \kappa_i} \xi(W) = \lim_{h \rightarrow 0} \frac{1}{h} (\xi(W + h \kappa_i) - \xi(W))$  exist in  $\hat{F}$  for all  $i \in I(r)$  and the sum converges in  $H$ .

(d) For  $W \in \Omega$ , set

$$\nabla_H \xi(W) := \nabla_G \xi(W) + \nabla_{W_0} \xi(W).$$

**Finite dimensional divergence** Let  $\xi \equiv \xi(W)$  be an  $F \equiv \mathbb{R}^{n \cdot d}$ -valued random variable. Let  $\nabla_{W_0}$  be either as in part (d) of Definition 1.1 or as in part (d) of Definition 1.2. Set

$$\nabla_{d,W_0} \xi(W) := \left( (\nabla_{W_0} (\xi(W)))_1, \dots, (\nabla_{W_0} (\xi(W)))_{n \cdot d} \right)^T.$$

**Definition 1.3** The *divergence* of  $\xi$  with respect to the coordinates of  $W_0$  is

$$\langle \mathbf{e}, \nabla_{d,W_0} \xi \rangle_F \equiv \sum_{j=1}^{n \cdot d} \left\langle e_j, \nabla_{W_0} \langle \xi, e_j \rangle_F \right\rangle_F.$$

**The spaces  $I^p$ ,  $E_{q,1}$ , and  $K_{q,1}$ ; flows belonging to  $\mathcal{F}_{p,1}(W)$ .**

**Definition 1.4** Let  $1 < p < \infty$  and  $I^p \equiv I^p(Q_\nu^{(m,r)})$  denote the set of all  $F$ -valued processes  $Y \equiv Y(W)$  such that

(i)  $Y(W) = Y_a(W) + Y_j(W)$  where

- (ii)  $Y_a(W)$  is an absolutely continuous process with  $j^{-1}Y_a \in L^p(\Omega, Q_\nu^{(m,r)}; H)$  and
- (iii)  $Y_j(W)$  is a cadlag pure jump process with  $\frac{1}{2}((Y_j)_{0-} + (Y_j)_0) = 0$  that is of locally finite variation on  $\mathcal{R}$ ,  $V_{-rt}^{(r-1)t}(Y_j) = \sum_{w \in \mathcal{R}} \sum_{i=1}^{n \cdot d} |((Y_j)_w - (Y_j)_{w-})_i| \in L^p(\Omega, Q_\nu^{(m,r)})$ .

**Definition 1.5** (a) Let  $\mathcal{F}_{p,1}(W) \equiv \mathcal{F}_{p,1}(W, Q_\nu^{(m,r)})$ ,  $1 < p < \infty$ , denote the set of all  $\Omega$ -valued random flows  $W^\rho \equiv W + g(\rho, W)$ ,  $\rho \in \mathbb{R}$ , on  $(\Omega, Q_\nu^{(m,r)})$ , with

- (i)  $\dot{g}(\rho, \cdot) \in jH := \{jh : h \in H\}$  given by the *mixed derivative*

$$\begin{aligned} \dot{g}(\rho, \cdot)(s) &= \frac{d^\pm}{d\rho} g(\rho, \cdot)(s) \\ &:= \frac{1}{2} \left( \frac{d^-}{d\rho} g(\rho, \cdot)(s) + \frac{d^+}{d\rho} g(\rho, \cdot)(s) \right), \quad s \in \mathcal{R}, \rho \in \mathbb{R} \quad Q_\nu^{(m,r)}\text{-a.e.} \end{aligned}$$

- (ii)  $j^{-1}\dot{g}(\rho, \cdot) \in L^p(\Omega, Q_\nu^{(m,r)}; H)$ ,  $\rho \in \mathbb{R}$ .

(b) Let  $Q$  be the number appearing in the definition of the density  $m$  and  $1/q + 1/Q < 1$ . Furthermore, let  $1 < p < \infty$  with  $1/p + 1/q = 1$ . Introduce

$$\begin{aligned} E_{q,1} &\equiv E_{q,1}(Q_\nu^{(m,r)}) \\ &:= \left\{ \varphi \in D_{q,1} : \frac{d^\pm}{d\sigma} \Big|_{\sigma=0} \varphi(W^\sigma) = \langle D\varphi(W), j^{-1}\dot{g}(0, W) \rangle_H \right. \\ &\quad \left. Q_\nu^{(m,r)}\text{-a.e. for all } (W^\rho)_{\rho \in \mathbb{R}} \in \mathcal{F}_{p,1}(W) \right\}. \end{aligned}$$

**Definition 1.6** Let  $Q$  be the number appearing in the definition of the density  $m$  and  $1/q + 1/Q < 1$ . Let  $K_{q,1} \equiv K_{q,1}(Q_\nu^{(m,r)})$  be the set of all processes  $Y$  for which

- (i)  $Y_s \in E_{q,1}$ ,  $s \in \mathcal{R}$ ,
- (ii)  $Q_\nu^{(m,r)}$ -a.e. we have  $D_t Y \in jH$  and  $t \rightarrow j^{-1}D_t Y$  has a right continuous version for which the linear span of  $\{j^{-1}D_t Y : t \in \mathcal{R}\}$  is dense in  $H$ . Furthermore,  $j^{-1}DY \in L^q(\Omega, Q_\nu^{(m,r)}; H \otimes H)$  and
- (iii) we have  $\langle j^{-1}DY, h \rangle_{H \rightarrow F} \in L^p(\Omega, Q_\nu^{(m,r)}; H)$ , and  $\langle DY, h \rangle_{H \rightarrow F} = j \langle j^{-1}DY, h \rangle_{H \rightarrow F}$  for all  $h \in L^p(\Omega, Q_\nu^{(m,r)}; H)$ ,  $1/p + 1/q = 1$ .

Let  $h \in H$ . It follows from (ii) that  $Q_\nu^{(m,r)}$ -a.e. we have

$$\langle |j^{-1}D.Y|, h \rangle_{H \rightarrow F} \in H, \quad \langle |j^{-1}DY|, h \rangle_{H \rightarrow F} \in H, \quad \langle j^{-1}DY, h \rangle_{H \rightarrow F} \in H.$$

**The process  $X$ .** Let  $A \equiv A(W)$  be a stochastic process with trajectories that belong to  $B_{b;\text{loc}}(\mathbb{R}; F)$  and are cadlag. Introduce

$$X := W + A.$$

This induces path wise a measurable map  $X = u(W) : \Omega \equiv C(\mathbb{R}; F) \rightarrow B_{b;\text{loc}}(\mathbb{R}; F)$  which we assume to be injective. The measure  $P_\mu := Q_\nu \circ u^{-1}$  is the law of  $X$ . Since  $Q_\nu(W_0 \in D^n) = 1$

it is reasonable to set  $A(W) \equiv 0$  whenever  $W_0 \notin D^n$ . Otherwise we will assume that  $X_s \in D^n$ ,  $s \in \mathbb{R}$ .

The subscript  $\mu$  indicates the distribution of  $X_0 = W_0 + A_0$ . We will use the symbol  $X$  when we primarily want to refer to  $X \equiv X(W)$  as a stochastic process, and we will use the symbol  $u \equiv u(W)$  in order to refer to the map  $u : C(\mathbb{R}; F) \rightarrow B_{b;\text{loc}}(\mathbb{R}; F)$ . Let us define  $\Omega^u := \{u(W) : W \in C(\mathbb{R}; F)\}$  and

$$P_\mu^{(m,r)} := Q_\nu^{(m,r)} \circ u^{-1} \quad \text{and} \quad P_\mu^{(m)} := Q_\nu^{(m)} \circ u^{-1}.$$

**Jumps of  $X$ .** Let  $\Delta X_w := X_w - X_{w-}$ . For  $W \in \Omega$  with  $W_0 = 0$  and  $X = u(W)$  let  $0 < \tau_1 \equiv \tau_1(X) < \tau_2 \equiv \tau_2(X) \dots$  denote the jump times of  $X$  on  $(0, \infty)$ , i.e., the times  $w$  for which  $\Delta X_w \neq 0$ . Accordingly, let  $\dots \tau_{-2} \equiv \tau_{-2}(X) < \tau_{-1} \equiv \tau_{-1}(X) \leq 0$  denote the jump times on  $(-\infty, 0]$ .

Based on this, let us introduce the numbering of the jumps of  $X = u(W)$  for general  $W_0 \in F$ .

**Definition 1.7** Suppose that for each  $k \in \mathbb{Z} \setminus \{0\}$  there is a map  $\tau_k : \Omega \equiv \{W \in \Omega : W_0 = 0\} \times F \rightarrow \mathbb{R}$  satisfying the following.

- (j) For each fixed  $W \in \Omega$  with  $W_0 = 0$  and  $k \in \mathbb{Z} \setminus \{0\}$  the map  $F \ni x \rightarrow \tau_k(u \circ (W + x\mathbb{I}))$  is *piecewise continuously differentiable* on  $F$ . That is, there exist finitely many mutually exclusive open sets  $F_1 \equiv F_1(k), F_2 \equiv F_2(k), \dots$  with piecewise  $C^1$ -boundary and  $\overline{F} = \bigcup_i \overline{F_i}$  such that

$$\tau_k(u \circ (W + \cdot \mathbb{I})) \Big|_{F_i} \in C_b^2(F_i), \quad i, k \in \mathbb{N}.$$

In addition, if  $x_0 \in \bigcup_i \partial F_i$  then there exist  $i_0 \equiv i_0(k)$  such that  $\tau_k(u \circ (W + \cdot \mathbb{I}))$  as well as  $\nabla_x \tau_k(u \circ (W + \cdot \mathbb{I}))$  can continuously be extended from  $F_{i_0}$  to  $x_0$ .

- (jj) For each  $W \in \Omega$  and  $X = u(W)$  the sequence  $\dots \tau_{-2} \equiv \tau_{-2}(X) < \tau_{-1} \equiv \tau_{-1}(X) < \tau_1 \equiv \tau_1(X) < \tau_2 \equiv \tau_2(X) \dots$  is the sequence of all *jump times of  $X$* .

For each  $W \in \Omega$  and  $X = u(W)$  let  $\tau_0 \equiv \tau_0(X) := \sup\{\tau_k : \tau_k \leq 0, k \in \mathbb{Z} \setminus \{0\}\}$ .

However keep in mind that  $\tau_{-1} \leq \tau_0 < \tau_1$  is not necessarily true. For every finite subinterval  $S$  of  $\mathbb{R}$  and all  $W \in \Omega$  with  $W_0 = 0$ , let us introduce

$$G(S; W) := \bigcup_{k,i} \overline{\{x \in \partial F_i(k) : \tau_k(u \circ (W + x\mathbb{I})) \in S\}}.$$

Let us furthermore define  $G(W) := \bigcup_S G(S; W)$  where the union is taken over all finite subinterval  $S$  of  $\mathbb{R}$  and stress that we have  $\nu(G(W)) = 0$ . Moreover, to simplify the notation we will use  $X_{\tau_k}(W)$ ,  $A_{\tau_k}(W)$ , etc. rather than  $X_{\tau_k \circ u}(W)$ ,  $A_{\tau_k \circ u}(W)$  etc.

**Remark. (1)** For given  $W \in \Omega$  with  $W_0 = 0$ , let  $\{G_i : i \in \mathbb{N}\}$  denote the set of all components of  $F \setminus G(W)$ . It is a consequence of Definition 1.7 that, for given  $W \in \Omega$  with  $W_0 = 0$  and  $x_0 \in D^n$  there exists a *time re-parametrization*  $\sigma(s, x) \equiv \sigma(s, x; W, x_0)$ ,  $s \in \mathbb{R}$ ,  $x \in F$ , with respect to  $W$  and  $x_0$  such that we have the following.

- (i)

$$\sigma \Big|_{\mathbb{R} \times G_i} \in C_b^2(\mathbb{R} \times G_i), \quad i \in \mathbb{N}.$$

- (ii) For  $x_1 \in \bigcup_i \partial G_i$  there exist  $i_1$  independent of  $x_0$  such that  $\sigma(s, \cdot)$  as well as  $\nabla_x \sigma(s, \cdot)$  can continuously be extended from  $G_{i_1}$  to  $x_1$ ,  $s \in \mathbb{R}$ .
- (iii)  $\sigma(s, x_0) = s$  for all  $s \in \mathbb{R}$ .
- (iv) For every  $x \in F$ ,  $\sigma(\cdot, x) \in C^1(\mathbb{R}; \mathbb{R})$  and, coordinate wise,

$$\frac{\partial}{\partial s} \sigma(s, x) > 0, \quad s \in S.$$

- (v)  $\sigma(\tau_k \circ u(W + x_0 \mathbb{1}), x) = \tau_k \circ u(W + x \mathbb{1})$ ,  $x \in F$ ,  $k \in \mathbb{Z} \setminus \{0\}$ .

Keeping property (v) of Remark (1) in mind, the next condition (jjj) says that jumps of parallel trajectories in  $\Omega$  have to be compatible with each other in a certain sense. This condition is trivially satisfied if, for example, parallel trajectories in  $\Omega$  generate identical jump times for  $X$ . In fact, the following condition (jjj) is crucial if parallel trajectories in  $\Omega$  do not necessarily generate identical jump times for  $X$ . Condition (jv) has been discussed in the motivating part of this section.

**Definition 1.7 continued.** For  $W \in \Omega$  with  $W_0 = 0$  and  $x_0 \in D^n$  we assume that there exists a time re-parametrization  $\sigma(s, x) \equiv \sigma(s, x; W, x_0)$ ,  $s \in \mathbb{R}$ ,  $x \in D^n$ , with respect to  $W$  and  $x_0$  such that (i)-(v) of Remark (1) and we have the following.

- (jjj) For every finite subinterval  $S$  of  $\mathbb{R}$  there is an  $\varepsilon \equiv \varepsilon(S) > 0$  such that for all  $k \in \mathbb{Z} \setminus \{0\}$  with  $\tau_k \circ u(W + x_0 \mathbb{1}) \in S$  we have

$$\begin{aligned} & \sigma(\tau_k \circ u(W + x \mathbb{1}) + \delta, x) - \sigma(\tau_k \circ u(W + x_0 \mathbb{1}), x) \\ &= \tau_k \circ u(W + x \mathbb{1}) + \delta - \tau_k \circ u(W + x_0 \mathbb{1}) \end{aligned}$$

for all  $|\delta| < \varepsilon$  and  $x \in D^n$ .

- (jv)  $\nabla_{W_0} \tau_k \circ u(W)$  is orthogonal in  $F$  to  $\Delta(A_{\tau_k}(W) - Y_{\tau_k}(W))$ ,  $k \in \mathbb{Z} \setminus \{0\}$ .

**Assumptions on  $X$ .** Let us post a list of assumptions on  $X$  which we will make use of in different stages of the paper. Again, let  $Q$  be the number appearing in the definition of the density  $m$ .

- (1)  $A$  has on  $\bigcup_m \{\pi_m W : W \in \Omega\}$  a local spatial gradient. That is,
  - (i) for all  $W \in \{\pi_m W : W \in \Omega\}$  with  $W_0 = 0$ ,  $x \in F$ ,  $s \in \mathbb{R}$ , and  $k \equiv k(W, x) = \max\{l \in \mathbb{Z} \setminus \{0\} : \tau_l(u(W + x \mathbb{1})) \leq s\}$  the gradient  $\nabla_x A_s(W + x \mathbb{1})$  exists and is bounded and continuous on some neighborhood  $U_{s,x} \subset F_i \times \mathbb{R} \equiv F_i(k) \times \mathbb{R}$  of  $(s, x)$  whenever  $x \in F_i$  and  $s > \tau_k(u(W + x \mathbb{1}))$ .
  - (ii) If  $x \in \bigcup_{i'} \partial F_{i'}(k)$  and  $s > \tau_k(u(W + x \mathbb{1}))$  then, with  $i_0$  given by (j), there exists a neighborhood  $U_s \times V_x$  of  $(s, x)$  such that the gradient  $\nabla_x A_s(W + x \mathbb{1})$  is well-defined, bounded, and continuous on  $(U_s \times V_x) \cap (\mathbb{R} \times F_{i_0}) \equiv (U_s \times V_x) \cap (\mathbb{R} \times F_{i_0}(k))$  and can continuously be extended from  $U_s \times F_{i_0}$  to  $U_s \times \{x\}$ .
  - (iii) If  $s = \tau_k(u(W + x \mathbb{1}))$  and  $x \in F_i(k)$  or  $x \in \partial F_{i_0}(k) \equiv \partial F_i(k)$  in the sense of the previous paragraph, then there exists a neighborhood  $U_{s,x}$  of  $(s, x)$  such that the gradient  $\nabla_x A_s(W + x \mathbb{1})$  is well-defined, bounded, and continuous on  $U_{s,x} \cap \{(v, y) \in \mathbb{R} \times F_i : \tau_{k(W,x)}(u(W + y \mathbb{1})) \leq v < \tau_{k(W,x)+1}(u(W + y \mathbb{1}))\}$  and can continuously be extended from this set to  $(s, x)$ .

- (iv) The bound on the gradients  $\nabla_x A_s(W + x\mathbb{1})$  is uniform for all  $W \in \bigcup_m \{\pi_m V : V \in \Omega, V_0 = 0\}$  and  $(s, x) \in S \times F$  where  $S$  is any finite subinterval of  $\mathbb{R}$ .

In this way,  $\nabla_{W_0} A_s(W)$  is well-defined for all  $W \in \bigcup_m \{\pi_m V : V \in \Omega\}$  and  $s \in \mathbb{R}$ . Condition (1) is rather sophisticated. In simplified situations we may use the following condition.

- (1')  $u_s \in D_{q,1} \equiv D_{q,1}(Q_\nu^{(m,r)}), s \in \mathbb{R}$ , for some  $q$  with  $1/q + 1/Q < 1$ .

We remark that for the well-definiteness of  $u_s \in D_{q,1}$  we only need  $1/q + 1/Q \leq 1$ , cf. Definition 1.1. However, for example, in Lemma 3.2 (f) below we need  $1/q + 1/Q < 1$ . For the next condition we recall the fact that the right continuous version of a Radon-Nikodym derivative with respect to the time is, whenever it exists, nothing but the right derivative. We will use the symbol “'” for this.

- (2) (i) For all  $W \in \Omega$  and  $A \equiv A(W)$ ,  $A = A^1 + A^2$ .  $A^1$  is a continuous process and  $A^2$  is a pure jump process with  $A_{\tau_1-}^2 = A_{\tau_1-}^2 = 0$  such that, for fixed time  $s \in \mathbb{R}$ ,  $A_s^2$  jumps only on  $Q_\nu^{(m)}$ - and  $Q_\nu$ -zero sets.  
Furthermore, on  $\bigcup_m \{(\pi_m W)_{\cdot+w} : W \in \Omega, w \in [0, \frac{1}{2^m} \cdot t)\}$ , the jumps of  $A^2$  do not accumulate at finite time and  $A^1$  possesses a Radon-Nikodym derivative with cadlag version  $A^{1'} = A'$ .  
(ii) Moreover,

$$\nabla_{W_0} A'_s(W) := \nabla_x|_{x=W_0} A'_s(W - W_0\mathbb{1} + x\mathbb{1}).$$

exists and is bounded on  $\bigcup_m \{(\pi_m W)_{\cdot+w} : W \in \Omega, w \in [0, \frac{1}{2^m} \cdot t)\}$  uniformly for all  $s$  belonging to any finite subinterval of  $\mathbb{R}$ . The same holds for the right continuous version of  $A_-$ .

- (iii) For  $W \in \bigcup_m \{(\pi_m V)_{\cdot+w} : V \in \Omega, w \in [0, \frac{1}{2^m} \cdot t)\}$  with  $W_0 = 0$  and  $k \in \mathbb{Z} \setminus \{0\}$  the map  $F \ni x \rightarrow \Delta A_{\tau_k}(W + x\mathbb{1})$  is piecewise continuously differentiable on  $F$ . That is  $\Delta A_{\tau_k}(W + \cdot\mathbb{1})$  satisfies (j) of the definition of the jump times with  $\Delta A_{\tau_k}(W + \cdot\mathbb{1})$  instead of  $\tau_k(W + \cdot\mathbb{1})$ .  
(3) (i) There exists  $Y \equiv Y(W)$  such that  $X \equiv X(W)$  is a *temporally homogeneous* function of  $W \in \Omega$  in the sense that for  $W^u := W_{\cdot+u} + (A_u(W) - Y_u(W))\mathbb{1}$  we have

$$W^0 = W, \quad A_0((W^u)^v) = A_0(W^{u+v}),$$

and

$$X_{\cdot+v}(W) = X(W^v), \quad u, v \in \mathbb{R}.$$

In the subsequent part (ii) we use the fact that, according to part (i) of this condition, the jump times of  $u(W^v)$  are the jump times of  $X = u(W)$  minus  $v$ . For a subset  $S \subset \mathbb{R}$ , we define  $S \ominus v = \{s \in \mathbb{R} : s + v \in S\}$ .

- (3) (ii) For every finite subinterval  $S$  of  $\mathbb{R}$  and  $W \in \{\pi_m V : V \in \Omega\}$  with  $W_0 \notin G(S; W - W_0\mathbb{1})$  there exists  $\varepsilon > 0$  such that for all  $v \in S$  the Euclidean distance of  $W_0^v$  and  $G(S \ominus v; W^v - W_0^v\mathbb{1})$  is at least  $\varepsilon$ .

- (iii) Furthermore,  $\sup_{s,W} |Y_s(W)| < \infty$  and  $Y(W) \equiv 0$  if  $W_0 \notin D^n$  and  $Y$  has on  $\bigcup_m \{\pi_m W : W \in \Omega\}$  a local spatial gradient (cf. (1)). In addition,  $Y$  has all the properties of  $A$  mentioned in (2). In particular, the set of the jump times of  $Y$  is a subset of the set of the jump times of  $A$ .

The following condition describes the notion of continuity on  $\Omega$  we are going to use.

- (4) (i)  $\Omega \ni W_n \xrightarrow{n \rightarrow \infty} W \in \Omega$  relative to the topology of coordinate wise uniform convergence on compact subsets of  $\mathbb{R}$  and  $W_0 \notin G(S; W - W_0 \mathbb{1})$  imply

$$A_v(W_n) \xrightarrow{n \rightarrow \infty} A_v(W) \quad \text{and} \quad Y_v(W_n) \xrightarrow{n \rightarrow \infty} Y_v(W)$$

and

$$\nabla_{W_0} A_v(W_n) - \nabla_{W_0} Y_v(W_n) \xrightarrow{n \rightarrow \infty} \nabla_{W_0} A_v(W) - \nabla_{W_0} Y_v(W)$$

for all  $v \in S \setminus \{\tau_k(X) : k \in \mathbb{Z} \setminus \{0\}\}$  and any finite subinterval  $S \subset \mathbb{R}$ . Furthermore,

$$\Delta A_{\tau_k}(W_n) - \Delta Y_{\tau_k}(W_n) \xrightarrow{n \rightarrow \infty} \Delta A_{\tau_k}(W) - \Delta Y_{\tau_k}(W), \quad k \in \mathbb{Z} \setminus \{0\}.$$

- (ii)

$$A(W_n + x \mathbb{1}) - Y(W_n + x \mathbb{1}) - (A(W + x \mathbb{1}) - Y(W + x \mathbb{1})) \xrightarrow{n \rightarrow \infty} 0$$

in  $L^1(S; F)$  uniformly in  $x \in F$  for all finite subintervals  $S \subset \mathbb{R}$ .

Instead of condition (4) the following will be used in particular in Section 2 below.

- (4')  $\Omega \ni W_n \xrightarrow{n \rightarrow \infty} W \in \Omega$  relative to the topology of coordinate wise uniform convergence on compact subsets of  $\mathbb{R}$  implies  $X_s(W_n) \equiv u_s(W_n) \xrightarrow{n \rightarrow \infty} X_s \equiv u_s(W)$  for all  $s \in \mathbb{R} \setminus \{\tau_k(X) : k \in \mathbb{Z} \setminus \{0\}\}$ .

**Remark. (2)** According to conditions (1) and (3)  $A$  and  $Y$  have on  $\{\pi_m V : V \in \Omega\}$  a local spatial gradient. Since by conditions (2) and (3),  $A_{\tau_k}^2(W + \cdot \mathbb{1})$  and  $Y_{\tau_k}^2(W + \cdot \mathbb{1})$  are for  $W \in \{\pi_m V : V \in \Omega\}$  with  $W_0 = 0$  and  $k \in \mathbb{Z} \setminus \{0\}$  piecewise continuously differentiable on  $F$ ,  $A^1$  and  $Y^1$  have on  $\{\pi_m V : V \in \Omega\}$  also a local spatial gradient.

To ease the formulation of Theorem 1.11 below one could also require the following.

- (4'') For all  $\bigcup_m \{\pi_m V : V \in \Omega\} \ni W_n \xrightarrow{n \rightarrow \infty} W \in \Omega$  relative to the topology of coordinate wise uniform convergence on compact subsets of  $\mathbb{R}$  with  $W_0 \notin G(S; W - W_0 \mathbb{1})$ , and all  $s \in S \setminus \{\tau_k(X) : k \in \mathbb{Z} \setminus \{0\}\}$  where  $X = u(W)$  there exist the limit

$$\lim_{n \rightarrow \infty} \nabla_{W_0} A_s^1(W_n) - \nabla_{W_0} Y_s^1(W_n) =: \nabla_{W_0} A_s^1(W) - \nabla_{W_0} Y_s^1(W)$$

where  $S$  is any finite subinterval of  $\mathbb{R}$ .

Assuming (4''), the piecewise continuous differentiability of  $\Delta A$  and  $\Delta Y$ , cf. conditions (2) and (3), together with condition (4) (i) define  $\nabla_{W_0} \Delta A_{\tau_k} - \nabla_{W_0} \Delta Y_{\tau_k}$ ,  $k \in \mathbb{Z} \setminus \{0\}$ ,  $Q_\nu$ -a.e.

In order to formulate the next condition, introduce

$$f(s) := \begin{cases} \exp \left\{ -1 / (1 - s^2) \right\}, & s \in (-1, 1), \\ 0, & s \in \mathbb{R} \setminus (-1, 1), \end{cases}$$

and  $c_n := \left(\int_{\mathbb{R}} f(ns) ds\right)^{-1}$  as well as  $c_{F,n} := \left(\int_F f(n|x|) dx\right)^{-1}$ . Define the mollifier functions  $g_n(s) := c_n \cdot f(ns) \mathbf{e} = nc_1 \cdot f(ns) \mathbf{e}$ ,  $s \in \mathbb{R}$ , and

$$\gamma_n(x) := c_{F,n^3} \cdot f(n^3|x|) \mathbf{e}, \quad x \in F, \quad n \in \mathbb{N}.$$

For the motivation of the order  $n^3$  in the definition of  $\gamma_n$  we refer to the text above (3.87) in Subsection 3.3. In addition, set

$$A_s(\cdot, \gamma_n)(W) := \int_F \langle A_s(W + x\mathbf{1}), \gamma_n(x) \rangle_{F \rightarrow F} dx$$

$s \in \mathbb{R}$ ,  $n \in \mathbb{N}$ . Likewise define  $Y_s(\cdot, \gamma_n)$ . Furthermore, let  $\kappa_i := \int_0^\cdot H_i(u) du$ ,  $i \in \mathbb{N}$ .

- (5) (i) For all  $r \in \mathbb{N}$  and all  $s \in \mathcal{R} \equiv \mathcal{R}(r)$  there exists  $h \equiv h_s(r) \in H \equiv H^{(r)}$  such that for all  $W, w \in jH$  with  $w_0 = 0$ , and all  $\varepsilon \in (-1, 1)$ ,

$$|A_s(W + \varepsilon w) - A_s(W)| \leq |\langle w', h_s \rangle_{L^2}| \cdot \max_j |A_s(W + \varepsilon e_j \mathbf{1}) - A_s(W)|$$

as well as

$$|Y_s(W + \varepsilon w) - Y_s(W)| \leq |\langle w', h_s \rangle_{L^2}| \cdot \max_j |Y_s(W + \varepsilon e_j \mathbf{1}) - Y_s(W)|$$

Furthermore  $\sup_{s \in \mathcal{R}} \|h_s\|_{L^2}^2 < \infty$ .

- (ii) Furthermore, for all  $r \in \mathbb{N}$ , all  $n \in \mathbb{N}$ , and all  $s \in \mathcal{R}$ ,  $A_s(\cdot, \gamma_n)(W)$  and  $Y_s(\cdot, \gamma_n)(W)$  are two times Fréchet differentiable on  $W \in jH \equiv jH^{(r)}$ .

For an  $F$ -valued random variable  $\xi$  we will use the symbol  $D_F \xi(W)$  to denote the Fréchet derivative of  $\xi$  at  $W \in jH$  if it exists. In this case,

$$D_F \xi(W) = \nabla_H \xi(W) \equiv \sum_{i \in I(r)} \frac{\partial \xi(W)}{\partial \kappa_i} (H_i, 0) + \sum_{j'} \frac{\partial \xi(W)}{\partial \lambda_{j'}} (0, e_{j'}) \in H.$$

Furthermore, we will use the notation

$$(D_{F,1})_s \xi(W) := \sum_{i \in I(r)} \frac{\partial \xi(W)}{\partial \kappa_i} \cdot H_i(s), \quad s \in \mathcal{R},$$

if  $\xi$  is on  $W \in jH$  Fréchet differentiable. For the second order Fréchet derivative we will use the symbol  $D_F^2 \xi(W)$ .

- (5) (iii) For all  $W \in jH$ , the gradients  $\nabla_G A_s(\cdot, \gamma_n)(W) \equiv (\nabla_G)_r A_s(\cdot, \gamma_n)(W)$  as well as  $\nabla_G Y_s(\cdot, \gamma_n)(W) \equiv (\nabla_G)_r Y_s(\cdot, \gamma_n)(W)$  are continuously differentiable with respect to the variable  $r \in \mathcal{R}$ .

**Remarks.** (on conditions (1)-(4)) **(3)** It follows from condition (3) that

$$X_v = X_0(W^v) = W_v + (A_v(W) - Y_v(W)) + A_0(W^v)$$

which implies

$$Y_v(W) = A_0(W^v), \quad v \in \mathbb{R}.$$

Furthermore,  $Y_v(W^u) = A_0((W^u)^v) = A_0(W^{u+v}) = Y_{u+v}(W)$  which means that we have in addition to condition (3) also

$$Y_{\cdot+v}(W) = Y(W^v), \quad v \in \mathbb{R}.$$

Moreover,

$$\begin{aligned} A_v(W^u) - A_0(W^u) &= X_v(W^u) - X_0(W^u) - ((W^u)_v - (W^u)_0) \\ &= X_v(W^u) - X_0(W^u) - (W_{u+v} - W_u) \\ &= X_{u+v}(W) - X_u(W) - (W_{u+v} - W_u) \\ &= A_{u+v}(W) - A_u(W), \quad u, v \in \mathbb{R}. \end{aligned}$$

It follows now from condition (3) that  $W(\cdot)$  possesses the *flow property*,

$$\begin{aligned} (W^u)^v &= (W^u)_{\cdot+v} + (A_v(W^u) - Y_v(W^u)) \mathbb{1} \\ &= W_{\cdot+u+v} + (A_u(W) - Y_u(W)) \mathbb{1} + (A_v(W^u) - Y_{u+v}(W)) \mathbb{1} \\ &= W_{\cdot+u+v} + (A_u(W) - A_0(W^u) + A_v(W^u)) \mathbb{1} - Y_{u+v}(W) \mathbb{1} \\ &= W_{\cdot+u+v} + (A_{u+v}(W) - Y_{u+v}(W)) \mathbb{1} \\ &= W^{u+v}, \quad u, v \in \mathbb{R}. \end{aligned}$$

(4) If  $Y_v(W) = Y_0(W) = A_0(W)$ ,  $v \in \mathbb{R}$ ,  $W \in \Omega$ , then condition (3) reads with  $W^u := W_{\cdot+u} + (A_u(W) - A_0(W)) \mathbb{1}$ ,  $u \in \mathbb{R}$ , as

$$A_0(W^v) = A_0(W), \quad X_{\cdot+v}(W) = X(W^v), \quad v \in \mathbb{R}.$$

If  $A_0(W) = 0$ ,  $W \in \Omega$ , then condition (3) (i) reads as simple as

$$X_{\cdot+v} = X(W_{\cdot+v} + A_v \mathbb{1}), \quad v \in \mathbb{R}.$$

(5) By condition (2), for all  $m \in \mathbb{N}$  there exists  $Q_\nu^{(m)}$ -a.e. a representation of  $A$  of the form

$$A_s(W) - A_0(W) = \int_0^s b(w, X(W)) dw + \sum_{w \in (0, s]} \Delta X_w(W), \quad s \in \mathbb{R}.$$

In case of  $A_0(W) = 0$ ,  $W \in \Omega$ , and  $G(W - W_0 \mathbb{1}) = \emptyset$  temporal homogeneity is under condition (2) and the first line of condition (4) (i) equivalent to the following.

(3') We have

$$b(s+v, X(W)) = b(s, X(W^v))$$

and

$$\Delta X_{s+v}(W) = \Delta X_s(W^v), \quad s, v \in \mathbb{R}, \quad Q_\nu^{(m)}\text{-a.e.}, \quad m \in \mathbb{N}.$$

(6) Let  $W \in \bigcup_m \{\pi_m V : V \in \Omega\}$ . Because of  $A(W + x\mathbb{1}) \equiv 0$  whenever  $W_0 + x \notin D^n$  and  $A_s(W + x\mathbb{1}) = X_s(W + x\mathbb{1}) - W_s - x$  and  $X_s \in D^n$  where  $D$  is a bounded domain,  $A_s(W + x\mathbb{1})$  is bounded on  $(s, x) \in S \times F$  for any finite subinterval  $S \subset \mathbb{R}$ .

It follows from condition (1) (iv) that both

$$\int \left\langle A_s(W + y\mathbb{1}), \gamma_n(y - x) \right\rangle_{F \rightarrow F} dy \quad \text{and} \quad \nabla_x \int \left\langle A_s(W + y\mathbb{1}), \gamma_n(y - x) \right\rangle_{F \rightarrow F} dy$$

are bounded on  $(s, x) \in S \times F$  for any finite subinterval  $S \subset \mathbb{R}$ . According to condition (4) (i) these relations hold even for all  $W \in \Omega$ .

By condition (3), the same holds for  $Y$ . Because of condition (3) we have for  $s = v + w$  the identity

$$(W + y\mathbb{1})^w = W_{.+w} + \left( y + A_w(W + y\mathbb{1}) - Y_w(W + y\mathbb{1}) \right) \mathbb{1}$$

and furthermore

$$\begin{aligned} A_v((W + y\mathbb{1})^w) &= X_v((W + y\mathbb{1})^w) - (W + y\mathbb{1})_v^w \\ &= X_s(W + y\mathbb{1}) - W_s - \left( y + A_w(W + y\mathbb{1}) - Y_w(W + y\mathbb{1}) \right) \\ &= A_s(W + y\mathbb{1}) - A_w(W + y\mathbb{1}) + Y_w(W + y\mathbb{1}). \end{aligned}$$

Thus, both

$$\int \left\langle A_v(W_{.+w} + y\mathbb{1}), \gamma_n(y - x) \right\rangle_{F \rightarrow F} dy$$

and

$$\nabla_x \int \left\langle A_v(W_{.+w} + y\mathbb{1}), \gamma_n(y - x) \right\rangle_{F \rightarrow F} dy$$

are bounded on  $(v, w, x) \in S \times S \times F$  for any finite subinterval  $S \subset \mathbb{R}$ . Similarly, corresponding relations for  $Y$  can be verified.

(7) Recall the existence of  $\nabla_{W_0} A'$  as well as  $\nabla_{W_0} Y'$  in the sense of condition (2) and the piecewise continuous differentiability of  $\Delta A$  and  $\Delta Y$ , cf. conditions (2) and (3). These properties yield the existence and the right continuity of  $s \rightarrow \nabla_{W_0} A_s^1(W) + \sum_{\{k \geq 1: \tau_k \circ u \leq s\}} \nabla_{W_0} \Delta A_{\tau_k}$  as well as  $s \rightarrow \nabla_{W_0} Y_s^1(W) + \sum_{\{k \geq 1: \tau_k \circ u \leq s\}} \nabla_{W_0} \Delta Y_{\tau_k}$  and implies

$$\nabla_{W_0} A_s = \nabla_{W_0} A_s^1(W) + \sum_{\{k \geq 1: \tau_k \circ u \leq s\}} \nabla_{W_0} \Delta A_{\tau_k}$$

as well as

$$\nabla_{W_0} Y_s = \nabla_{W_0} Y_s^1(W) + \sum_{\{k \geq 1: \tau_k \circ u \leq s\}} \nabla_{W_0} \Delta Y_{\tau_k}$$

for all, without loss of generality,  $s > \tau_{-1} \circ u$  and  $W \in \{(\pi_m V)_{.+w} : V \in \Omega, w \in [0, \frac{1}{2^m} \cdot t)\}$  such that  $W_0 \notin G(S; W - W_0\mathbb{1})$ . For the consequence of this remark to the formulation of Theorem 1.11 below, recall condition (4").

**The spaces  $D_{q,1}^u$ ,  $E_{q,1}^u$ , and  $K_{q,1}^u$ ; flows belonging to  $\mathcal{F}_{p,1}(X)$ .** For the rest of this subsection, let  $X = u(W)$  be a bijection  $C(\mathbb{R}; F) \rightarrow C(\mathbb{R}; F)$ . Although this implies  $\Omega = \Omega^u$  we will use the symbol  $\Omega^u$  for a better indication.

**Definition 1.8** Let  $\mathcal{C}(\Omega^u)$  denote the set of all cylindrical functions of Definition 1.1 with  $W$  replaced by  $X$ . Let  $Q$  be the number from the definition of the density  $m$  and let  $1/q + 1/Q \leq 1$ . Let  $\mathbb{D}^u$  be that version of  $\mathbb{D}$  that acts on functions  $\psi$  with arguments  $X = u(W)$  rather than  $W$ . In particular, suppose that the operator  $(D^u, \mathcal{C}) := (j^* \circ \mathbb{D}^u, \mathcal{C}(\Omega^u))$  is closable on  $L^q(\Omega^u, P_\mu^{(m,r)}; H)$ .

(a) Following the lines of Definition 1.1 and replacing there  $W$  by  $X$ , we introduce the space  $D_{q,1}^u \equiv D_{q,1}^u(P_\mu^{(m,r)})$  as the closure of  $(D^u, \mathcal{C}(\Omega^u)) := (j^* \circ \mathbb{D}^u, \mathcal{C}(\Omega^u))$  on  $L^q(\Omega^u, P_\mu^{(m,r)}; H)$ .

(b) If no ambiguity is possible we will also use the symbol  $(D^u, D_{q,1}^u)$  to denote the (vectors of) gradients of functions of type  $\Omega^u \rightarrow F$ .

(c) Let  $\psi \in D_{q,1}^u$  be of type  $\Omega^u \rightarrow \mathbb{R}$ . If  $\psi$  is for  $X \in \Omega^u$  of the form  $\psi(X - X_0 \mathbb{1}, X_0)$  and the  $n \cdot d$ -dimensional vector of the directional derivatives of  $\psi$  in the directions of the components of  $X_0$  exists in the sense of differentiation in  $L^q(\Omega^u, P_\mu^{(m,r)}; F)$  then let it be denoted by  $\nabla_{X_0} \psi \equiv \nabla_{X_0} \psi(X)$ .

(d) Let  $\xi \equiv \xi(X)$  be an  $F \equiv \mathbb{R}^{n \cdot d}$ -valued random variable. Set

$$\nabla_{d,X_0} \xi(X) := \left( (\nabla_{X_0} (\xi(X)))_1, \dots, (\nabla_{X_0} (\xi(X)))_{n \cdot d} \right)^T.$$

The *divergence* of  $\xi$  with respect to the coordinates of  $X_0$  is

$$\langle \mathbf{e}, \nabla_{d,X_0} \xi \rangle_F \equiv \sum_{j=1}^{n \cdot d} \left\langle e_j, \nabla_{X_0} \langle \xi, e_j \rangle_F \right\rangle_F.$$

**Definition 1.9** (a) Let  $1 < p < \infty$  and  $\mathcal{F}_{p,1}(X) \equiv \mathcal{F}_{p,1}(X, P_\mu^{(m,r)})$  denote the set of all  $\Omega^u$ -valued random flows  $X^\rho \equiv X + g(\rho, X)$ ,  $\rho \in \mathbb{R}$ , on  $(\Omega^u, P_\mu^{(m,r)})$  with

(i)  $\dot{g}(\rho, \cdot) \in jH := \{jh : h \in H\}$  given by

$$\dot{g}(\rho, \cdot)(s) = \frac{d^\pm}{d\rho} g(\rho, \cdot)(s), \quad s \in \mathcal{R}, \quad \rho \in \mathbb{R} \quad P_\mu^{(m,r)}\text{-a.e.}$$

(ii)  $j^{-1} \dot{g}(\rho, \cdot) \in L^p(\Omega^u, P_\mu^{(m,r)}; H)$ ,  $\rho \in \mathbb{R}$ .

(b) Let  $Q$  be the number from the definition of the density  $m$  and let  $1/q + 1/Q < 1$  as well as  $1/p + 1/q = 1$ . Furthermore, let  $(X^\rho)_{\rho \in \mathbb{R}} \in \mathcal{F}_{p,1}(X)$ . Introduce

$$\begin{aligned} E_{q,1}^u &\equiv E_{q,1}^u(P_\mu^{(m,r)}) \\ &:= \left\{ \varphi \in D_{q,1}^u : \frac{d^\pm}{d\sigma} \Big|_{\sigma=0} \varphi(X^\sigma) = \langle D^u \varphi(X), j^{-1} \dot{g}(0, X) \rangle_H \right. \\ &\quad \left. P_\mu^{(m,r)}\text{-a.e. for all } (X^\rho)_{\rho \in \mathbb{R}} \in \mathcal{F}_{p,1}(X) \right\}. \end{aligned}$$

**Definition 1.10** Let  $Q$  be the number from the definition of the density  $m$  and  $1/q + 1/Q < 1$ . Let  $K_{q,1}^u \equiv K_{q,1}^u(P_\mu^{(m,r)})$  be the set of all processes  $Y$  for which

(i)  $Y_s \in E_{q,1}^u$ ,  $s \in \mathcal{R}$ ,

(ii)  $P_\mu^{(m,r)}$ -a.e. we have  $D_t^u Y \in jH$ ,  $t \in \mathcal{R}$ . Furthermore,  $j^{-1} D^u Y \in L^q(\Omega^u, P_\mu^{(m,r)}; H \otimes H)$ ,

- (iii)  $\langle j^{-1}D^u Y, h \rangle_{H \rightarrow F} \in L^p(\Omega^u, P_\mu^{(m,r)}; H)$ , and  $\langle D^u Y, h \rangle_{H \rightarrow F} = j \langle j^{-1}D^u Y, h \rangle_{H \rightarrow F}$  for all  $h \in L^p(\Omega^u, P_\mu^{(m,r)}; H)$ ,  $1/p + 1/q = 1$ .

**Remarks. (8)** (on Definitions 1.8-1.10) When working with one of the spaces  $D_{q,1}^u$ ,  $E_{q,1}^u$ , or  $K_{q,1}^u$  we will always suppose that the operator  $(D^u, \mathcal{C}) := (j^* \circ \mathbb{D}^u, \mathcal{C}(\Omega^u))$  is closable on  $L^q(\Omega^u, P_\mu^{(m,r)}; H)$ .

**(9)** (on Definitions 1.5 and 1.9) Let  $\varphi \in \mathcal{C}(\Omega)$  be a cylindrical function in the sense of Definition 1.1 (a), i. e.,  $\varphi(W) := f(\langle H, W \rangle, \langle e, W \rangle)$  with  $f \in C_b^1(\mathbb{R}^{\#I(m,r)+n \cdot d})$ ,  $\langle e, W \rangle$  denoting the vector of all  $\langle e_j, W_0 \rangle_F$ ,  $j \in \{1, \dots, n \cdot d\}$ , and  $\langle H, W \rangle$  being the vector of all  $\langle H_i, dW \rangle_{L^2}$ ,  $i \in I(m, r)$ ,  $W \in C(\mathbb{R}; F)$ . By Definition 1.5 (a)

$$\left. \frac{d^\pm}{d\sigma} \right|_{\sigma=0} \langle H_i, dW^\sigma \rangle_{L^2} = \langle H_i, d\dot{g}(0, W) \rangle_H$$

because of the particular structure of  $H_i$ ,  $W^\sigma = W + g(\sigma, W)$ , and  $(W^\rho)_{\rho \in \mathbb{R}} \in \mathcal{F}_{p,1}(W)$ . Furthermore,

$$\left. \frac{d^\pm}{d\sigma} \right|_{\sigma=0} \langle e_j, W_0^\sigma \rangle_F = \langle e_j, \dot{g}(0, W)(0) \rangle_F.$$

Let  $f_i$  denote the first order derivative of  $f$  relative to  $i \in I(m, r)$  and let  $f_j$  be the first order derivative of  $f$  relative to  $j \in \{1, \dots, n \cdot d\}$ . The last two relations imply  $\varphi \in E_{1,q}$  since

$$\begin{aligned} \left. \frac{d^\pm}{d\sigma} \right|_{\sigma=0} \varphi(W^\sigma) &= \left. \frac{d^\pm}{d\sigma} \right|_{\sigma=0} f(\langle H, W^\sigma \rangle, \langle e, W^\sigma \rangle) \\ &= \sum_{i \in I(m,r)} f_i(\langle H, W \rangle, \langle e, W \rangle) \cdot \left. \frac{d^\pm}{d\sigma} \right|_{\sigma=0} \langle H_i, dW^\sigma \rangle_{L^2} \\ &\quad + \sum_{j=1}^{n \cdot d} f_j(\langle H, W \rangle, \langle e, W \rangle) \cdot \left. \frac{d^\pm}{d\sigma} \right|_{\sigma=0} \langle e_j, W_0^\sigma \rangle_F \\ &= \langle D\varphi(W), j^{-1}\dot{g}(0, W) \rangle_H \quad Q_\nu^{(m,r)}\text{-a.e. for all } (W^\rho)_{\rho \in \mathbb{R}} \in \mathcal{F}_{p,1}(W). \end{aligned}$$

Similarly,  $\psi \in E_{1,q}^u$  if  $\psi \in \mathcal{C}(\Omega^u)$ .

### 1.3 Main result

The main purpose of the paper is to specify a class of processes  $X \equiv X(W)$  and a class of distributions  $\nu$  for which the Radon-Nikodym densities

$$\omega_{-t}(W) := \frac{Q_\nu(dW^{-t})}{Q_\nu(dW)} \quad \text{and} \quad \rho_{-t}(X) := \frac{P_\mu(dX_{-t})}{P_\mu(dX)}, \quad t \in \mathbb{R},$$

exist and to give a representation of these densities. Let  $E_\nu$  denote the expectation with respect to the measure  $Q_\nu$  and let  $|\cdot|_i$  be the absolute value of the  $i$ -th coordinate.

**Theorem 1.11** Assume (1)-(5). (a) For  $t \in \mathbb{R}$  the density  $\omega_{-t}$  exists and we have  $Q_\nu$ -a.e.

$$\omega_{-t}(W) = \frac{m(X_{-t} - Y_{-t})}{m(W_0)} \cdot \prod_{i=1}^{n \cdot d} \left| e + \nabla_{d, W_0} A_{-t} - \nabla_{d, W_0} Y_{-t} \right|_i. \quad (1.1)$$

If, in addition, (4'') is satisfied then with  $I_{-t} := ((-t) \wedge \tau_{-1}, (-t) \vee \tau_{-1}]$ ,  $b = 1$  if  $\tau_{-1} < -t$ , and  $b = -1$  if  $\tau_{-1} \geq -t$  we have the representation

$$\omega_{-t}(W) = \frac{m(X_{-t} - Y_{-t})}{m(W_0)} \times \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} (A_{-t}^1 - Y_{-t}^1) + b \cdot \sum_{\{k: \tau_k \circ u \in I_{-t}\}} \nabla_{d, W_0} \Delta (A_{\tau_k} - Y_{\tau_k}) \right|_i.$$

(b) For  $t \in \mathbb{R}$  the density  $\rho_{-t}$  exists and we have  $P_\mu$ -a.e.

$$\rho_{-t}(X) = \frac{m(X_{-t} - Y_{-t} \circ u^{-1})}{m(u_0^{-1})} \cdot \prod_{i=1}^{n \cdot d} \left| \nabla_{d, W_0} X_{-t} - \nabla_{d, W_0} Y_{-t} \circ u^{-1} \right|_i. \quad (1.2)$$

If  $A^2 = 0$ , i. e., if  $X$ ,  $A$ , and  $Y$  are continuous processes, then the conditions under which we have (1.1) and (1.2) simplify as follows.

**Theorem 1.12** *Assume the following.*

- (i)  $A$  has on  $\bigcup_m \{\pi_m W : W \in \Omega\}$  a local spatial gradient. That is, for all  $W \in \{\pi_m V : V \in \Omega\}$  with  $W_0 = 0$ ,  $x \in F$ , and  $s \in \mathbb{R}$ , the gradient  $\nabla_x A_s(W + x\mathbf{I})$  exists such that we have (1) (iv). Furthermore, suppose (1').
- (ii) On  $\{\pi_m W : W \in \Omega\}$ ,  $A$  possesses a Radon-Nikodym derivative with cadlag version  $A'$ . Furthermore, we have (2) (ii).
- (iii) We have (3) (i) and (iii) (without jump times) as well as  $Y_s \in D_{q,1}(Q_\nu^{(m,r)})$ ,  $s \in \mathbb{R}$ , for  $q$  defined in (1').
- (iv)  $\bigcup_m \{\pi_m V : V \in \Omega\} \ni W_n \xrightarrow{n \rightarrow \infty} W \in \Omega$  relative to the topology of coordinate wise uniform convergence on compact subsets of  $\mathbb{R}$  implies

$$A_v(W_n) \xrightarrow{n \rightarrow \infty} A_v(W) \quad \text{and} \quad Y_v(W_n) \xrightarrow{n \rightarrow \infty} Y_v(W)$$

and

$$\nabla_{W_0} A_v(W_n) - \nabla_{W_0} Y_v(W_n) \xrightarrow{n \rightarrow \infty} \nabla_{W_0} A_v(W) - \nabla_{W_0} Y_v(W)$$

for all  $v \in S$  and any finite subinterval  $S \subset \mathbb{R}$ . Also suppose (4) (ii).

- (v) For all  $r \in \mathbb{N}$ , all  $n \in \mathbb{N}$ , and all  $s \in \mathcal{R}$ ,  $A_s(W)$  and  $Y_s(W)$  are two times Fréchet differentiable on  $W \in jH$ . Furthermore, for all  $W \in jH$ , the gradients  $\nabla_G A_s(W) \equiv (\nabla_G)_r A_s(W)$  as well as  $\nabla_G Y_s(W) \equiv (\nabla_G)_r Y_s(W)$  are continuous with respect to the variable  $r \in \mathcal{R}$ .

Then we have (1.1) and (1.2).

We stress that the technical conditions on the density  $m$  formulated in the beginning of Subsection 1.2 are crucial in the proofs of Theorem 1.11 and Theorem 1.12. A significant relaxation of the conditions on  $m$  is possible once we have determined the representations (1.1) and (1.2) for  $\omega_{-t}(W)$  and  $\rho_{-t}(X)$  under the conditions of Subsection 1.2. This two-step procedure is the reason for presenting the relaxation in a separate corollary.

**Corollary 1.13** *Suppose (1.1) for some  $\bar{m}$  satisfying the conditions on the Lebesgue density of  $W_0$  formulated in the beginning of Subsection 1.2. Let  $m$  be a strictly positive, everywhere on  $D^n$  defined real function which is on  $D^n$  integrable respect to  $\lambda_F$ . Suppose also  $\int_{D^n} m \, dx = 1$  and let  $Q_\nu$  be defined accordingly. Then, for  $Q_\nu$  and  $P_\mu := Q_\nu \circ u^{-1}$ , relations (1.1) and (1.2) hold true.*

Proof. Let  $F$  be a bounded measurable function on  $\Omega$ . Set  $\bar{\nu} := \bar{m} \, dx$  and let  $Q_{\bar{\nu}}$  be defined accordingly. Furthermore, let  $\bar{\omega}$  be the expression defined in (1.1) with  $m$  replaced by  $\bar{m}$ . We have

$$\begin{aligned} \int F \, dQ_\nu &= \int F(W) \cdot \frac{m(W_0)}{\bar{m}(W_0)} \, dQ_{\bar{\nu}} \\ &= \int F(W^{-t}) \cdot \frac{m(W_0^{-t})}{\bar{m}(W_0^{-t})} \cdot \bar{\omega}_{-t}(W) \, dQ_{\bar{\nu}} \\ &= \int F(W^{-t}) \cdot \frac{m(X_{-t} - Y_{-t})}{\bar{m}(X_{-t} - Y_{-t})} \cdot \bar{\omega}_{-t}(W) \cdot \frac{\bar{m}(W_0)}{m(W_0)} \, dQ_\nu \\ &= \int F(W^{-t}) \cdot \omega_{-t}(W) \, dQ_\nu. \end{aligned}$$

□

We continue with two corollaries of Theorem 1.11 and Theorem 1.12 for which we pay particular attention to condition (1) (iv) and (4) (i) and, respectively, (i) and (iv) of Theorem 1.12. The first one is the following disintegration formula for the measure  $P_\mu$ . For this let  $\xi_X := \{X_{-t} : t \in \mathbb{R}\}$ ,  $u^{-1}(X) \in \Omega$ , and  $\mathcal{X} := \{\xi_X : u^{-1}(X) \in \Omega\}$ . For  $A \in \mathcal{F}$  set  $\mathcal{A} := \{\xi_X : X = u(W), W \in A\}$ . Endow  $\mathcal{X}$  with the sub- $\sigma$  algebra  $\mathcal{F}_\mathcal{X}$  of  $u \circ \mathcal{F}$  generated by all such sets  $\mathcal{A}$ . Furthermore, define the measure  $\Gamma$  on  $(\mathcal{X}, \mathcal{F}_\mathcal{X})$  by  $\Gamma(\mathcal{A}) := P_\mu(\{X : \text{there exists } \xi \in \mathcal{A} \text{ such that } X \in \xi\})$ .

**Corollary 1.14** *Suppose the hypotheses of Theorem 1.11 or Theorem 1.12 and let  $m$  be as in Corollary 1.13. For  $\Gamma$ -a.e.  $\xi \in \mathcal{X}$  there exists a measure  $\gamma_\xi$  on  $\xi$  endowed with the trace- $\sigma$  algebra  $\mathcal{F}_\xi$  of  $u \circ \mathcal{F}$  to  $\xi$  such that for  $A \in u \circ \mathcal{F}$  and  $A_\xi := \xi \cap A$*

$$P_\mu(A) = \int_{\xi \in \mathcal{X}} \gamma_\xi(A_\xi) \, \Gamma(d\xi).$$

For  $\Gamma$ -a.e.  $\xi \in \mathcal{X}$  and  $X \in \xi$ , we have

$$\gamma_\xi(\{X_{+\tau_k(X)} : k \in \mathbb{Z} \setminus \{0\}\}) = 0.$$

Furthermore, for  $\Gamma$ -a.e.  $\xi \in \mathcal{X}$ ,  $X \in \xi$ ,  $-t \in \mathbb{R} \setminus \{\tau_k(X) : k \in \mathbb{Z} \setminus \{0\}\}$ , and  $X^1 = X_{-t}$  we have

$$\frac{\gamma_\xi(dX^1)}{\gamma_\xi(dX)} = \rho_{-t}(X)$$

where  $\rho$  is given by (1.2).

An immediate but useful consequence of Corollary 1.14 as well as Theorem 1.11 or Theorem 1.12 is the following one.

**Corollary 1.15** *Suppose the hypotheses of Theorem 1.11 or Theorem 1.12 and let  $m$  be as in Corollary 1.13. Let  $\mathcal{A}$  be an index set. Assume that*

(i) *for every  $\alpha \in \mathcal{A}$ , there is a set  $\Omega_\alpha \in \mathcal{F}$  such that  $\bigcup_{\alpha \in \mathcal{A}} \Omega_\alpha = \Omega$  and  $\alpha_1 \neq \alpha_2$  implies  $\Omega_{\alpha_1} \cap \Omega_{\alpha_2} = \emptyset$  and*

(ii)  *$u^{-1} \circ X \in \Omega_\alpha$  implies  $u^{-1} \circ X_{\cdot-t} \in \Omega_\alpha$  for all  $t \in \mathbb{R}$ ,  $\alpha \in \mathcal{A}$ .*

*Let  $t(\alpha) : \mathcal{A} \rightarrow \mathbb{R}$  be a bounded map and define the random time  $\tau^\alpha \equiv \tau^\alpha(X)$  on  $(\Omega, \mathcal{F})$  by  $\tau^\alpha \circ u := t(\alpha)$  on  $\Omega_\alpha$ ,  $\alpha \in \mathcal{A}$ . Then we have*

$$P_\mu(-\tau^\alpha \text{ is a jump time for } X) = 0$$

and

$$\frac{P_\mu(dX_{\cdot-\tau^\alpha})}{P_\mu(dX)} = \rho_{-\tau^\alpha}(X) \quad P_\mu\text{-a.e.}$$

where  $\rho$  is given by (1.2).

The restriction to bounded domains  $D$  is due to techniques used in the proofs. A possibility to establish absolute continuity under time shift of trajectories of processes with unbounded state space comes with the following proposition. For this let  $X \equiv X(W) = W + A$  be a stochastic process with  $A_0 = 0$  and state space  $\check{D}^n$  where  $\check{D} \subseteq \mathbb{R}^d$  is a possibly unbounded domain. Let  $D \subset \mathbb{R}^d$  be a bounded domain in the above setting and  $g : \check{D}^n \rightarrow D^n$  be a bijection. Define

$$\hat{W} := g(W_0)\mathbb{1} + W - W_0\mathbb{1}, \quad W \in \Omega, \quad W_0 \in \check{D}^n,$$

and

$$\check{W} := g^{-1}(W_0)\mathbb{1} + W - W_0\mathbb{1}, \quad W \in \Omega, \quad W_0 \in D^n.$$

Furthermore, let  $\hat{A}(W) := g \circ X(\check{W}) - W$  for  $W \in \Omega$  with  $W_0 \in D^n$ . This yields

$$\hat{X}(W) := g \circ X(\check{W}) = W + \hat{A}(W), \quad W \in \Omega, \quad W_0 \in D^n.$$

The formulation and the proof of the following proposition does not take into consideration the distribution of  $W_0$ .

**Proposition 1.16** *Assume the following.*

(i)  *$X = W + A$  where  $W$  is a two-sided Brownian motion with the law of  $W_0$  supported by  $\overline{\check{D}^n}$  and  $X : \Omega \rightarrow B_{b,\text{loc}}(\mathbb{R}; F)$  is a measurable map.*

(ii)  *$A_0 = 0$ .*

(iii) *With  $W^v := W_{\cdot+v} + A_v(W)\mathbb{1}$  we have  $X(W^v) = X_{\cdot+v}(W)$ ,  $v \in \mathbb{R}$ .*

*Then  $\hat{A}_0(W) = 0$  for all  $W \in \Omega$  such that  $W_0 \in D^n$  and with  $W^{\wedge v} := W_{\cdot+v} + \hat{A}_v(W)\mathbb{1}$  we have*

$$\hat{X}(W^{\wedge v}) = \hat{X}_{\cdot+v}(W), \quad v \in \mathbb{R}.$$

Proof. Let  $W \in \Omega$  with  $W_0 \in D^n$ . By (i) and (ii), we have  $\hat{A}_0(W) = g(X_0(\check{W})) - W_0 = g(\check{W}_0 + A_0(\check{W})) - W_0 = g(\check{W}_0) - W_0 = 0$ . In addition,

$$\widetilde{W_0^v} = g^{-1}(W_0^v) = g^{-1}(W_v + \hat{A}_v(W)) = g^{-1}(\hat{X}_v(W)) = X_v(\check{W}) = \check{W}_0^v$$

which implies

$$\widetilde{W^v} = (\check{W})^v \equiv \check{W}^v, \quad v \in \mathbb{R}.$$

With (iii) we obtain

$$\hat{X}(W^v) = g \circ X.(\widetilde{W^v}) = g \circ X.(\check{W}^v) = g \circ X_{+v}(\check{W}) = \hat{X}_{+v}(W), \quad v \in \mathbb{R}.$$

□

In fact, based on (i) and (iii) of Proposition 1.16, we have verified condition (3) (i) for  $\hat{X}$ . In words, this says that for  $A = 0$  temporal homogeneity is invariant under bijective space transformation. For the absolute continuity under time shift of trajectories for  $\hat{X}$  check now the remaining conditions of Theorem 1.11 or Theorem 1.12, respective Corollary 1.13 relative to  $\hat{X}$ . An immediate consequence is absolute continuity under time shift of trajectories for  $X$ .

## 2 Flows and Logarithmic Derivative Relative to $X$ under Orthogonal Projection

In this section we are primarily interested in the analysis of the process  $X = u(W)$  where we just focus on conditions (1') and (4') of Subsection 1.2. In particular, we do not refer to  $X = W + A$ . In Subsection 2.1, we will introduce elements of the calculus on orthogonal projections of  $W$ . Subsections 2.2 and 2.3 will be dedicated to the analysis of related flows and logarithmic derivatives.

In order to prove Theorem 1.11 we will apply the following approximation theorem.

**Theorem 2.1** (*Specification of Theorem 3.1 in [6], Theorem 3 in [10]*) Suppose that  $M$  is a separable metric space,  $\mu$  is a probability measure defined on the Borel  $\sigma$ -algebra over  $M$ , and  $f_n : M \rightarrow M$ ,  $n \in \mathbb{N}$ , is a sequence of measurable maps. Assume that the following conditions are satisfied:

- (i) For every  $n \in \mathbb{N}$  the measure  $\mu \circ f_n^{-1}$  is absolutely continuous with respect to the measure  $\mu$ .
- (ii) The sequence of densities  $d\mu \circ f_n^{-1}/d\mu$ ,  $n \in \mathbb{N}$ , is uniformly integrable.
- (iii)  $f_n \xrightarrow{n \rightarrow \infty} f$  in the measure  $\mu$  for some  $f : M \rightarrow M$ .

Then  $\mu \circ f^{-1}$  is absolutely continuous with respect to the measure  $\mu$ . If  $d\mu \circ f_n^{-1}/d\mu \xrightarrow{n \rightarrow \infty} p$  in the measure  $\mu$  then  $p = d\mu \circ f^{-1}/d\mu$ .

## 2.1 Elements of the analysis on orthogonal projections of Brownian paths

Starting point is the Lévy-Ciesielsky construction of  $W$ . Accordingly, there are independent  $N(0, 1)$ -distributed random variables  $\xi_1, \xi_2, \dots$  such that  $W_s = W_0 + \sum_{i=1}^{\infty} \xi_i \cdot \int_0^s H_i(u) du$ ,  $s \in \mathbb{R}$ , where the sum converges uniformly in  $s$  on finite subintervals of  $\mathbb{R}$ ,  $Q_\nu$ -a.e. It holds that  $\xi_i = \langle H_i, dW \rangle_{L^2}$  so that we get the Lévy-Ciesielsky representation

$$W_s = W_0 + \sum_{i=1}^{\infty} \langle H_i, dW \rangle_{L^2} \cdot \int_0^s H_i(u) du, \quad s \in \mathbb{R}.$$

Under the measure  $Q_\nu^{(m,r)}$  we shall consider

$$W_s = W_0 + \sum_{i \in I(m,r)} \langle H_i, dW \rangle_{L^2} \cdot \int_0^s H_i(u) du, \quad s \in \mathbb{R}, \quad (2.1)$$

and  $j^{-1}W = q_{m,r}j^{-1}W = \sum_{i \in I(m,r)} \langle H_i, dW, W_0 \rangle_{L^2} \cdot H_i \in H$ . In the proof of Theorem 1.11 in Section 3 we will also consider similar projections under the measure  $Q_\nu^{(m)}$ .

**The integral  $\int \langle V, d\dot{W} \rangle_F$  under the measure  $Q_\nu^{(m,r)}$ .** For  $-rt \leq u < v \leq (r-1)t$  and  $V : \mathcal{R} \rightarrow F$  cadlag define

$$\begin{aligned} & \int_{w=u}^v \langle V, dH_i \rangle_F \\ &:= \sum_{w \in (u,v)} \frac{1}{2} \langle V_{w-} + V_w, \Delta H_i(w) \rangle_F + \frac{1}{2} \langle V_u, \Delta H_i(u) \rangle_F + \frac{1}{2} \langle V_{v-}, \Delta H_i(v) \rangle_F, \end{aligned}$$

$i \in I(m, r)$ , and similarly the integral  $\int_{w=u}^v \langle V, dH_i \rangle_{F \rightarrow F}$ . Furthermore, introduce

$$\int_{w=u}^v \langle V, d\dot{W} \rangle_F := \sum_{i \in I(m,r)} \langle H_i, dW \rangle_{L^2} \cdot \int_{w=u}^v \langle V, dH_i \rangle_F \quad Q_\nu^{(m,r)}\text{-a.e.} \quad (2.2)$$

For  $i, j \in I(m, r)$ , we have

$$\langle H_i, dH_j \rangle_{L^2} + \langle H_j, dH_i \rangle_{L^2} = 0.$$

It follows that

$$\begin{aligned} \sum_{i \in I(m,r)} \langle H_i, dW \rangle_{L^2} \cdot \langle H_i, d\dot{W} \rangle_{L^2} &= \sum_{i,j \in I(m,r)} \langle H_i, dW \rangle_{L^2} \langle H_j, dW \rangle_{L^2} \cdot \langle H_i, dH_j \rangle_{L^2} \\ &= 0 \quad Q_\nu^{(m,r)}\text{-a.e.} \end{aligned} \quad (2.3)$$

**The process  $X$  and the map  $u$  under the measure  $Q_\nu^{(m,r)}$ .** Let

$$\begin{aligned} \kappa_i &:= \int_0^\cdot H_i(v) dv, \quad i \in I(m, r), \\ \lambda_j &:= e_j \mathbb{I}(\cdot), \quad j \in \{1, \dots, n \cdot d\}. \end{aligned}$$

**Lemma 2.2** (a) Let  $Q$  be the number used in the definition of the density  $m$  and  $1/q+1/Q \leq 1$ . Let  $\varphi \in D_{q,1}$  with respect to the measure  $Q_\nu^{(m,r)}$  and  $\varphi_{n'} \in \mathcal{C}$ ,  $n' \in \mathbb{N}$ , be a sequence with  $\varphi_{n'} \xrightarrow{n' \rightarrow \infty} \varphi$  in  $D_{q,1}$ . Choosing a subsequence if necessary  $D\varphi$  is  $Q_\nu^{(m,r)}$ -a.e. a finite dimensional object for which

$$\begin{aligned} D\varphi(W) &= \lim_{n' \rightarrow \infty} D\varphi_{n'}(W) \\ &= \lim_{n' \rightarrow \infty} \sum_{i \in I(m,r)} \frac{\partial \varphi_{n'}(W)}{\partial \kappa_i} \cdot (H_i, 0) + \lim_{n' \rightarrow \infty} \sum_{j=1}^{n \cdot d} \frac{\partial \varphi_{n'}(W)}{\partial \lambda_j} \cdot (0, e_j) \\ &= \sum_{i \in I(m,r)} \langle D\varphi, (H_i, 0) \rangle_H \cdot (H_i, 0) + \sum_{j=1}^{n \cdot d} \langle D\varphi, (0, e_j) \rangle_H \cdot (0, e_j). \end{aligned}$$

In particular,  $\varphi = \varphi \circ \pi_{m,r}$  and  $D\varphi = D\varphi \circ \pi_{m,r}$   $Q_\nu^{(m,r)}$ -a.e.

(b) Assume condition (4') of Subsection 1.2. We have

$$\lim_{m \rightarrow \infty} X_t(\pi_m W) = X_t, \quad t \in \mathbb{R} \setminus \{\tau_i(X) : i = \pm 1, \pm 2, \dots\} \quad Q_\nu\text{-a.e.}$$

(c) Let  $\xi \equiv \xi(W)$  be an  $\hat{F}$ -valued random variable with  $\hat{F}$  being a metric space. Assume that  $W_n \xrightarrow{n \rightarrow \infty} W$  relative to the topology of coordinate wise uniform convergence on compact subsets of  $\mathbb{R}$  implies  $\xi(W_n) \xrightarrow{n \rightarrow \infty} \xi(W)$ . Let  $\psi \in C_b(\hat{F})$  or  $\psi \in C(\hat{F})$  and  $\sup_{r \in \mathbb{N}} |\psi(\xi(\pi_{m,r} W))| \in L^1(\Omega, Q_\nu^{(m)})$ . Then

$$\lim_{r \rightarrow \infty} \int \psi(\xi(W)) Q_\nu^{(m,r)}(dW) = \int \psi(\xi(W)) Q_\nu^{(m)}(dW).$$

Proof. The representation of  $D\varphi$  in part (a) follows from Definition 1.1. By the Lévy-Ciesielsky construction of  $W$  and the above definition of  $D\varphi$ ,  $\varphi$  and  $D\varphi$  are independent of  $\langle H_i, dW \rangle_{L^2}$ ,  $i \notin I(m,r)$ . In other words,  $\varphi = \varphi \circ \pi_{m,r}$  and  $D\varphi = D\varphi \circ \pi_{m,r}$   $Q_\nu^{(m,r)}$ -a.e.

Part (b) is an immediate consequence of (4'), cf. Subsection 1.2. For part (c) it is sufficient to recall that  $Q_\nu^{(m,r)} = Q_\nu \circ \pi_{m,r}^{-1}$ , relation (2.1), the corresponding representation of  $W$  under the measure  $Q_\nu^{(m)} = Q_\nu \circ \pi_m^{-1}$ , and condition (4'). In fact, it follows now that

$$\begin{aligned} \lim_{r \rightarrow \infty} \int \psi(\xi(W)) Q_\nu^{(m,r)}(dW) &= \lim_{r \rightarrow \infty} \int \psi(\xi(\pi_{m,r} W)) Q_\nu^{(m)}(dW) \\ &= \int \psi(\xi(W)) Q_\nu^{(m)}(dW). \end{aligned}$$

□

For  $s \in \mathcal{R} = [-rt, (r-1)t]$ , we have  $Du_s \in H$  by assumption (1') of Section 1 with  $Du_s = ((Du_s)_1, (Du_s)_2)$  and  $(Du_s)_1 \in L_{\text{loc}}^2(\mathbb{R}; F)$  and  $(Du_s)_2 \in F$   $Q_\nu^{(m,r)}$ -a.e. However,  $Q_\nu^{(m,r)}$ -a.e.,  $(Du_s)_1$  is, according to Lemma 2.2, supported by a subset of  $\mathcal{R}$ . So we will assume  $Q_\nu^{(m,r)}$ -a.e.,  $(Du_s)_1 \in L^2(\mathcal{R}; F)$  and  $(Du_s)_2 \in F$ .

Let  $\mathcal{M}^{f,s}(\mathcal{R}; F)$  denote the set of all  $F$ -valued finite signed measures on  $(\mathcal{R}, \mathcal{B}(\mathcal{R}))$ .

**Lemma 2.3** Let  $X = u(W)$  be a map  $C(\mathbb{R}; F) \rightarrow B_{b,\text{loc}}(\mathbb{R}; F)$ . Assume (1') of Section 1 and let  $\gamma : C(\mathcal{R}; F) \rightarrow \mathcal{M}^{f,s}(\mathcal{R}; F)$ . Assume furthermore that

(i) for every  $i \in I(m, r)$ ,  $\langle H_i, d\gamma \rangle_{L^2}$  is differentiable in direction of  $\kappa_i$ , that is

$$\langle H_i, d\gamma \rangle_{L^2} \in L^p(\Omega, Q_\nu^{(m,r)}) \quad \text{and} \quad \frac{\partial \langle H_i, d\gamma \rangle_{L^2}}{\partial \kappa_i} \text{ exists in } L^p(\Omega, Q_\nu^{(m,r)})$$

where  $1/p + 1/q = 1$  and  $q$  is the constant of assumption (1') of Section 1 and

(ii) for every  $i' \in \{1, \dots, n \cdot d\}$ ,  $\langle e_{i'}, d\gamma(0) \rangle_F$  is differentiable in direction of  $e_{i'}$ , that is

$$\langle e_{i'}, d\gamma(0) \rangle_F \in L^p(\Omega, Q_\nu^{(m,r)}) \quad \text{and} \quad \frac{\partial \langle e_{i'}, d\gamma(0) \rangle_F}{\partial e_{i'}} \text{ exists in } L^p(\Omega, Q_\nu^{(m,r)}).$$

(iii) The linear span of  $\{X_s : s \in \mathcal{R}\}$  is dense in  $L^q(\Omega, Q_\nu^{(m,r)}; F)$ .

Then  $\langle (Du_s)_1, d\gamma \rangle_{L^2 \rightarrow F} + \langle (Du_s)_2, d\gamma(0) \rangle_{F \rightarrow F} = 0$  for all  $s \in \mathcal{R}$   $Q_\nu^{(m,r)}$ -a.e. implies that

$$\langle H_i, d\gamma \rangle_{L^2} = \langle e_{i'}, d\gamma(0) \rangle_F = 0, \quad i \in I(m, r), \quad i' \in \{1, \dots, n \cdot d\}.$$

Proof. For  $i \in I(m, r)$ , let  $\beta^{(i,0)}$  denote the logarithmic derivative relative to the measure  $\langle H_i, d\gamma \rangle_{L^2} dQ_\nu^{(m,r)}$ . Moreover, for  $i' \in \{1, \dots, n \cdot d\}$ , let  $\beta^{(0,i')}$  denote the logarithmic derivative relative to the measure  $\langle e_{i'}, d\gamma(0) \rangle_F dQ_\nu^{(m,r)}$ . Both logarithmic derivatives exist by hypothesis. By Lemma 2.2 (a) we may write

$$\begin{aligned} & \int \langle ((Du_s)_1, d\gamma)_{L^2 \rightarrow F} + \langle (Du_s)_2, d\gamma(0) \rangle_{F \rightarrow F} \rangle dQ_\nu^{(m,r)} \\ &= \sum_{i \in I(m,r)} \int \langle Du_s, (H_i, 0) \rangle_{H \rightarrow F} \cdot \langle H_i, d\gamma \rangle_{L^2} dQ_\nu^{(m,r)} \\ & \quad + \sum_{i' \in \{1, \dots, n \cdot d\}} \int \langle Du_s, (0, e_{i'}) \rangle_{H \rightarrow F} \cdot \langle e_{i'}, d\gamma(0) \rangle_F dQ_\nu^{(m,r)} \\ &= - \sum_{\substack{i \in I(m,r) \\ i' \in \{1, \dots, n \cdot d\}}} \int u_s \cdot \left( \beta_{j(H_i, 0)}^{(i)} \cdot \langle H_i, d\gamma \rangle_{L^2} + \beta_{j(0, e_{i'})}^{(i')} \cdot \langle e_{i'}, d\gamma(0) \rangle_F \right) dQ_\nu^{(m,r)}. \end{aligned}$$

By  $\langle (Du_s)_1, d\gamma \rangle_{L^2 \rightarrow F} + \langle (Du_s)_2, d\gamma(0) \rangle_{F \rightarrow F} = 0$  for all  $s \in \mathcal{R}$   $Q_\nu^{(m,r)}$ -a.e. and assumption (iii), we have

$$\sum_{\substack{i \in I(m,r) \\ i' \in \{1, \dots, n \cdot d\}}} \left( \beta_{j(H_i, 0)}^{(i)} \cdot \langle H_i, d\gamma \rangle_{L^2} + \beta_{j(0, e_{i'})}^{(i')} \cdot \langle e_{i'}, d\gamma(0) \rangle_F \right) = 0 \quad Q_\nu^{(m,r)}\text{-a.e.} \quad (2.4)$$

In addition, it holds that

$$\begin{aligned} & \int \left\langle \left( \langle (Du_s)_1, d\gamma \rangle_{L^2 \rightarrow F} + \langle (Du_s)_2, d\gamma(0) \rangle_{F \rightarrow F} \right), W_v \right\rangle_{F \rightarrow F} dQ_\nu^{(m,r)} \\ & \quad + \sum_{\substack{i \in I(m,r) \\ i' \in \{1, \dots, n \cdot d\}}} \left( \int \left\langle \int_0^v H_i(w) dw, u_s \right\rangle_{F \rightarrow F} \cdot \langle H_i, d\gamma \rangle_{L^2} dQ_\nu^{(m,r)} \right. \\ & \quad \left. + \int \langle e_{i'}, u_s \rangle_{F \rightarrow F} \cdot \langle e_{i'}, d\gamma(0) \rangle_F dQ_\nu^{(m,r)} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{i \in I(m,r) \\ i' \in \{1, \dots, n \cdot d\}}} \left( \int \langle \langle Du_s, (H_i, 0) \rangle_{H \rightarrow F}, W_v \rangle_{F \rightarrow F} \cdot \langle H_i, d\gamma \rangle_{L^2} dQ_\nu^{(m,r)} \right. \\
&\quad + \int \langle \langle Du_s, (0, e_{i'}) \rangle_{H \rightarrow F}, W_v \rangle_{F \rightarrow F} \cdot \langle e_{i'}, d\gamma(0) \rangle_F dQ_\nu^{(m,r)} \\
&\quad + \int \langle \langle DW_v, (H_i, 0) \rangle_{H \rightarrow F}, u_s \rangle_{F \rightarrow F} \cdot \langle H_i, d\gamma \rangle_{L^2} dQ_\nu^{(m,r)} \\
&\quad \left. + \int \langle \langle DW_v, (0, e_{i'}) \rangle_{H \rightarrow F}, u_s \rangle_{F \rightarrow F} \cdot \langle e_{i'}, d\gamma(0) \rangle_F dQ_\nu^{(m,r)} \right) \\
&= - \int \langle u_s, W_v \rangle_{F \rightarrow F} \cdot \sum_{\substack{i \in I(m,r) \\ i' \in \{1, \dots, n \cdot d\}}} \left( \beta_{j(H_i, 0)}^{(i)} \cdot \langle H_i, d\gamma \rangle_{L^2} + \beta_{j(0, e_{i'})}^{(i')} \cdot \langle e_{i'}, d\gamma(0) \rangle_F \right) dQ_\nu^{(m,r)}.
\end{aligned}$$

By  $\langle (Du_s)_1, d\gamma \rangle_{L^2 \rightarrow F} + \langle (Du_s)_2, d\gamma(0) \rangle_{F \rightarrow F} = 0$  for all  $s \in \mathcal{R}$   $Q_\nu^{(m,r)}$ -a.e. and (2.4) we conclude now

$$\begin{aligned}
&\sum_{\substack{i \in I(m,r) \\ i' \in \{1, \dots, n \cdot d\}}} \int \left( \left\langle \int_0^v H_i(w) dw, u_s \right\rangle_{F \rightarrow F} \cdot \langle H_i, d\gamma \rangle_{L^2} \right. \\
&\quad \left. + \langle e_{i'}, u_s \rangle_{F \rightarrow F} \langle e_{i'}, d\gamma(0) \rangle_F \right) dQ_\nu^{(m,r)} = 0,
\end{aligned}$$

$s, v \in \mathcal{R}$ , which by assumption (iii) proves the lemma.  $\square$

## 2.2 Two flows associated with $X = u(W)$ , substitution law, and stochastic integral relative to $X$

We may write  $X \equiv X(W) := u(W)$ . Suppose we are given  $h$  with  $h \circ u : C(\mathbb{R}; F) \rightarrow B_{b; \text{loc}}(\mathbb{R}; F)$  and  $f(\rho, W)(s) : \mathbb{R} \times C(\mathbb{R}; F) \rightarrow C(\mathbb{R}; F)$  such that

$$\begin{aligned}
&\int_{\sigma=0}^{\rho} h \circ u(W + f(\sigma, W))(\cdot) d\sigma = u(W + f(\rho, W)) - u(W) \\
&= X(W + f(\rho, W)) - X(W), \quad \rho \in \mathbb{R}, \quad (2.5)
\end{aligned}$$

$$f(0, W) \equiv 0.$$

We notice that since  $X = u(W)$  is an injection  $\Omega \rightarrow \Omega^u$  and  $W^\rho, \rho \in \mathbb{R}$ , is a random flow on  $\Omega$ , because of

$$\begin{aligned}
X^{\rho+\sigma} &= u(W^{\rho+\sigma})u^{-1} = u(W^\rho \circ W^\sigma)u^{-1} \\
&= u(W^\rho)u^{-1} \circ u(W^\sigma)u^{-1} = X^\sigma \circ X^\rho, \quad \rho, \sigma \in \mathbb{R},
\end{aligned}$$

$X^\rho, \rho \in \mathbb{R}$ , is a random flow on  $\Omega^u$ . In this situation, there are two random flows in this equation,  $W^\rho := W + f(\rho, W)$  and  $X^\rho \equiv X^\rho(W) := u(W + f(\rho, W)), \rho \in \mathbb{R}$ . In Section 3 we will be interested in specifying the second one to the formal relation  $X^\rho := X_{\cdot+\rho}, \rho \in \mathbb{R}$ . In

order to prepare this we are going to develop elements of the related stochastic differential and integral calculus in this section.

**Assumptions for the remainder of Section 2.** In this section we shall assume that for  $Q_\nu^{(m,r)}$ -a.e.  $W \in \Omega \equiv C(\mathbb{R}; F)$

$$f(\rho, W) = f^0(\rho, W) + f(\rho, W)(0) \cdot \mathbf{1}, \quad \rho \in \mathbb{R},$$

where  $f^0(\rho, W) \in C_a(\mathbb{R}; F)$  with Radon-Nikodym derivative in  $L_{\text{loc}}^2(\mathbb{R}; F)$  and  $f^0(\rho, W)(0) = 0$ . Furthermore,  $f(\rho, \cdot)(0) \equiv f(\rho, W)(0)$ ,  $\rho \in \mathbb{R}$ , is a stochastic process that we abbreviate by

$$f(\rho, W)(0) = B_\rho(W).$$

In particular, we shall suppose that  $B \equiv B_\rho(W)$  as well as  $B_- \equiv B_{-\rho}(W)$  possess  $Q_\nu^{(m,r)}$ -a.e. a Radon-Nikodym derivative with respect to  $\rho \in \mathbb{R}$  with cadlag versions  $B'$  and  $B'_-$ . We recall that  $B'$  and  $B'_-$  are at the same time  $Q_\nu^{(m,r)}$ -a.e. the right derivatives of  $B$  and  $B_-$ , respectively. Thus the *mixed derivative*

$$\dot{B}_\rho := \frac{1}{2} \left( \frac{d^-}{d\rho} B_\rho + \frac{d^+}{d\rho} B_\rho \right), \quad \rho \in \mathbb{R},$$

is well-defined.

Let  $1/p + 1/q = 1$  and  $q$  be given by condition (1') of Section 1. Let “ $\cdot$ ” stand further on for differentiation with respect to  $\rho$  and assume in this and the next subsection the following (2.6) and (2.7). We shall suppose that, under the measure  $Q_\nu^{(m,r)}$ ,  $W^\rho$ ,  $\rho \in \mathbb{R}$ , is *transient* in the sense that

$$W^\sigma \neq W^\rho \text{ for } \sigma \neq \rho. \quad (2.6)$$

Let “ $\cdot$ ” indicate the right continuous version of the Radon-Nikodym derivative of  $f^0(\rho, \cdot)(s)$  with respect to the Lebesgue measure  $ds$ . We also shall assume that  $(f^0(\rho, \cdot))'$  is  $Q_\nu^{(m,r)}$ -a.e. *weakly mixed differentiable* in  $\rho \in \mathbb{R}$  in the sense that for some  $\dot{f}(\rho, \cdot) \in I^p$ ,  $\rho \in \mathbb{R}$ , and all test elements  $k = (g, x) \in H$  with  $g \in C(\mathcal{R}; F)$ ,  $x \in F$ , we have

$$\begin{aligned} & \left\langle g, d\dot{f}(\rho, W) \right\rangle_{L^2} + \left\langle x, \dot{f}(\rho, W)(0) \right\rangle_F \\ &= \frac{1}{2} \left( \frac{d^-}{d\rho} \left\langle g, df^0(\rho, W) \right\rangle_{L^2} + \frac{d^+}{d\rho} \left\langle g, df^0(\rho, W) \right\rangle_{L^2} \right) + \left\langle x, \dot{B}_\rho \right\rangle_F, \quad \rho \in \mathbb{R}. \end{aligned} \quad (2.7)$$

**Hypotheses for Subsection 2.2.** Let  $X = u(W)$  be a bijection  $C(\mathbb{R}; F) \rightarrow C(\mathbb{R}; F)$  and  $q$  given by condition (1') of Section 1. Assume that the operator  $(D^u, \mathcal{C}) := (j^* \circ \mathbb{D}^u, \mathcal{C}(\Omega^u))$  is closable on  $L^q(\Omega^u, P_\mu^{(m,r)}; H)$ . Suppose also (2.6). Let  $1/p + 1/q = 1$  and  $(W^\rho)_{\rho \in \mathbb{R}} \in \mathcal{F}_{p,1}(W)$  which implies

$$\dot{f}(\rho, W)(s) = \dot{f}^0(\rho, W)(s) + \dot{B}_\rho \cdot \mathbf{1}(s), \quad s \in \mathcal{R}, \quad \rho \in \mathbb{R}. \quad (2.8)$$

We mention that  $(W^\rho)_{\rho \in \mathbb{R}} \in \mathcal{F}_{p,1}(W)$  (cf. Definition 1.5 (a)) is because of (2.8) stronger than (2.7) together with  $\dot{f}(\rho, \cdot) \in I^p$ ,  $\rho \in \mathbb{R}$ .

Furthermore, we suppose  $u \circ W^\rho \in K_{q,1}$  and  $u^{-1} \circ X^\rho \in K_{q,1}^u$ ,  $\rho \in \mathbb{R}$ , where  $q$  is given by condition (1') of Section 1.

**Relationships among the flows  $(W^\rho)_{\rho \in \mathbb{R}}$  and  $(X^\rho)_{\rho \in \mathbb{R}}$  and between  $u$  and  $u^{-1}$ .** Let us abbreviate  $Du_s \equiv D(u(s))$ ,  $s \in \mathbb{R}$ . We observe that the formal relation (2.5) gets now, because of  $(W^\rho)_{\rho \in \mathbb{R}} \in \mathcal{F}_{p,1}(W)$  and  $u \circ W^\rho \in K_{q,1}$ ,  $\rho \in \mathbb{R}$ , (especially Definition 1.6 (i)), a sense by

$$\begin{aligned} h \circ (u \circ W^\rho)(W) &= \left\langle D(u \circ W^\rho)(W), j^{-1} \dot{f}(0, W) \right\rangle_{H \rightarrow F} \\ &= \frac{d^\pm}{d\sigma} \Big|_{\sigma=0} (u \circ W^\rho)(W^\sigma) = \frac{d^\pm}{d\sigma} \Big|_{\sigma=0} u(W^{\rho+\sigma})(W) \\ &= \frac{d^\pm}{d\rho} u(W^\rho)(W), \quad \rho \in \mathbb{R}, \quad \text{for } Q_\nu^{(m,r)}\text{-a.e. } W \in \Omega \end{aligned} \quad (2.9)$$

and, in particular,

$$h \circ u(W)(\cdot) = \left\langle Du(W), j^{-1} \dot{f}(0, W) \right\rangle_{H \rightarrow F} = \frac{d^\pm}{d\sigma} \Big|_{\sigma=0} u(W^\sigma). \quad (2.10)$$

Let  $\nabla_{W_0} u_t$  be the vector of the gradients of the components of  $X_t \equiv u_t(W) \equiv u_t(W - W_0 \mathbb{I}, W_0)$  relative to the components of  $W_0$  and let  $\nabla_{X_0} u_t^{-1}$  denote the vector of the gradients of the components of  $W_t \equiv u_t^{-1}(X) \equiv u_t^{-1}(X - X_0 \mathbb{I}, X_0)$  relative to the components of  $X_0$ ,  $t \in \mathcal{R}$ .

**Proposition 2.4** *Let  $X = u(W)$  be a bijection  $C(\mathbb{R}; F) \rightarrow C(\mathbb{R}; F)$ . Suppose condition (1') of Subsection 1.2. Let  $q$  be the number given by in this condition (1'). Suppose that the operator  $(D^u, \mathcal{C}) := (j^* \circ \mathbb{D}^u, \mathcal{C}(\Omega^u))$  is closable on  $L^q(\Omega^u, P_\mu^{(m,r)}; H)$ . Assume that the following holds.*

- (i)  $(W^\rho)_{\rho \in \mathbb{R}} \in \mathcal{F}_{p,1}(W)$ ,  $1/p + 1/q = 1$ . The flow  $(W^\rho)_{\rho \in \mathbb{R}}$  satisfies (2.6).
- (ii)  $u \circ W^\rho \in K_{q,1}$  and  $u^{-1} \circ X^\rho \in K_{q,1}^u$ ,  $\rho \in \mathbb{R}$ .
- (a) We have  $(u(W^\rho))_{\rho \in \mathbb{R}} = (X^\rho)_{\rho \in \mathbb{R}} \in \mathcal{F}_{p,1}(X)$ . In addition to (2.9),

$$\dot{f}(\rho, u^{-1}(X))(\cdot) = \left\langle D^u(u^{-1} \circ X^\rho)(X), j^{-1} h(X) \right\rangle_{H \rightarrow F}, \quad \rho \in \mathbb{R},$$

for  $P_\mu^{(m,r)}$ -a.e.  $X \in \Omega^u$ .

- (b) We have,  $Q_\nu^{(m,r)}$ -a.e.,

$$\mathbb{I}_{[0,s]}(\cdot) \cdot \mathbf{e} = \left\langle j^{-1}(D.u)_1, (D^u u_s^{-1}) \circ u \right\rangle_{H \rightarrow F} \quad \text{and} \quad \mathbf{e} = \left\langle j^{-1} \nabla_{W_0} u, (D^u u_s^{-1}) \circ u \right\rangle_{H \rightarrow F}$$

and,  $P_\mu^{(m,r)}$ -a.e.,

$$\mathbb{I}_{[0,s]}(\cdot) \cdot \mathbf{e} = \left\langle j^{-1}(D^u u^{-1})_1, (Du_s) \circ u^{-1} \right\rangle_{H \rightarrow F} \quad \text{and} \quad \mathbf{e} = \left\langle j^{-1} \nabla_{X_0} u^{-1}, (Du_s) \circ u^{-1} \right\rangle_{H \rightarrow F}.$$

- (c) For  $\psi \in E_{q,1}^u$  it holds that

$$D.\psi \circ u = \left\langle j^{-1} D.u, (D^u \psi) \circ u \right\rangle_{H \rightarrow F} \quad Q_\nu^{(m,r)}\text{-a.e.}$$

Furthermore,  $\langle D\psi \circ u, k \rangle_H \in L^1(\Omega, Q_\nu^{(m,r)})$  if  $k \in L^p(\Omega, Q_\nu^{(m,r)}; H)$ .

Proof. (a) By (i), the first part of (ii) (especially Definition 1.6 (i)), and by (2.9) we get,  $P_\mu^{(m,r)}$ -a.e.,

$$\frac{d^\pm}{d\rho} X^\rho = \frac{d^\pm}{d\rho} u(W^\rho) = \left\langle D(u \circ W^\rho)(W), j^{-1} \dot{f}(W, 0) \right\rangle_{H \rightarrow F} = h(X^\rho), \quad \rho \in \mathbb{R}.$$

Now (i) and the first part of (ii) (especially Definition 1.6 (iii)) imply  $h \circ u(W^\rho) \in jH$   $Q_\nu^{(m,r)}$ -a.e. and  $j^{-1}h \circ u(W^\rho) \in L^p(\Omega, Q_\nu^{(m,r)}; H)$ ,  $\rho \in \mathbb{R}$ .

In other words, we have  $h(X^\rho) \in jH$   $P_\mu^{(m,r)}$ -a.e. and  $j^{-1}h(X^\rho) \in L^p(\Omega^u, P_\mu^{(m,r)}; H)$ ,  $\rho \in \mathbb{R}$ . Furthermore, we have by hypothesis  $X^\rho = u(W^\rho) \in C(\mathbb{R}; F)$ ,  $\rho \in \mathbb{R}$ . We conclude  $(X^\rho)_{\rho \in \mathbb{R}} \in \mathcal{F}_{p,1}(X)$ . Now, by the second part of (ii),  $P_\mu^{(m,r)}$ -a.e.,

$$\begin{aligned} \dot{f}(\rho, u^{-1}(X))(\cdot) &= \frac{d^\pm}{d\rho} W^\rho = \frac{d^\pm}{d\rho} u^{-1}(u(W^\rho)) \\ &= \left\langle D^u(u^{-1} \circ X^\rho)(X), j^{-1}h(X) \right\rangle_{H \rightarrow F}, \quad \rho \in \mathbb{R}. \end{aligned}$$

(b) For the set of all flows  $W^\cdot \equiv (W^\rho)_{\rho \in \mathbb{R}} \in \mathcal{F}_{p,1}(W)$ , with (i) of Definition 1.5 it follows that,  $Q_\nu^{(m,r)}$ -a.e.,

$$H \subseteq \left\{ j^{-1} \dot{f}(0, \cdot) : W^\cdot \in \mathcal{F}_{p,1}(W) \right\} \quad (2.11)$$

by just considering the specific flows of type  $W^\rho := W + \rho \cdot jk$ ,  $\rho \in \mathbb{R}$ , where  $k \in H$ . In contrast to the formulation of the proposition, in the following, the “.” indicates the variables to be jointly integrated over. Also, one should read  $j^{-1}Du. \equiv ((Du.)', Du_0) \in H$ . Keeping (2.11) in mind, the first part of (b) follows from

$$\begin{aligned} \dot{f}(0, W)(s) &= \left\langle (D^u u_s^{-1}) \circ u, j^{-1} \left\langle Du., j^{-1} \dot{f}(0, W) \right\rangle_{H \rightarrow F} \right\rangle_{H \rightarrow F} \\ &= \left\langle (D^u u_s^{-1}) \circ u, \left\langle j^{-1} Du., j^{-1} \dot{f}(0, W) \right\rangle_{H \rightarrow F} \right\rangle_{H \rightarrow F} \\ &= \left\langle \left( \left\langle j^{-1}(Du.)_1, (D^u u_s^{-1}) \circ u \right\rangle_{H \rightarrow F}, \left\langle j^{-1} \nabla_{W_0} u, (D^u u_s^{-1}) \circ u \right\rangle_{H \rightarrow F} \right), j^{-1} \dot{f}(0, W) \right\rangle_{H \rightarrow F}. \end{aligned}$$

For the first equality sign, recall (a) and (2.10). For the second one, we refer to Definition 1.6 (iii). For the third one use Fubini's theorem together with Definition 1.6 (ii) as well as the remarks after Definition 1.6.

Assuming that there was an element  $k \in H$ ,  $k \neq 0$ , such that  $0 = \langle k, j^{-1}h(X) \rangle_H$  for all  $h(X) = \frac{d^\pm}{d\sigma} X^\sigma \Big|_{\sigma=0}$  on a set of positive  $P_\mu^{(m,r)}$ -measure where  $X^\rho = u(W^\rho)$ ,  $\rho \in \mathbb{R}$ , and  $W^\cdot \in \mathcal{F}_{p,1}(W)$ , from (2.9) it would follow that

$$\begin{aligned} 0 &= \left\langle k(\cdot), j^{-1} \left\langle Du., j^{-1} \dot{f}(0, W) \right\rangle_{H \rightarrow F} \right\rangle_H \\ &= \left\langle \left( \left\langle j^{-1}(Du.)_1, k(\cdot) \right\rangle_{H \rightarrow F}, \left\langle j^{-1} \nabla_{W_0} u, k \right\rangle_{H \rightarrow F} \right), j^{-1} \dot{f}(0, W) \right\rangle_H. \end{aligned}$$

By Definition 1.6 (ii) and (2.11) the linear span of

$$\left\{ j^{-1}h(X) = j^{-1} \frac{d^\pm}{d\sigma} X^\sigma \Big|_{\sigma=0} : X^\cdot = u(W^\cdot), W^\cdot \in \mathcal{F}_{p,1}(W) \right\}$$

is therefore  $P_\mu^{(m,r)}$ -a.e. dense in  $H$ . The second part of (b) is now by (a) and (2.10) a consequence of

$$\begin{aligned} h(X)(s) &= \left\langle (D.u_s) \circ u^{-1}, j^{-1} \left\langle D^u u^{-1}, j^{-1} h(X) \right\rangle_{H \rightarrow F} \right\rangle_{H \rightarrow F} \\ &= \left\langle \left( \left\langle j^{-1} (D^u u^{-1})_1, (D.u_s) \circ u^{-1} \right\rangle_{H \rightarrow F}, \right. \right. \\ &\quad \left. \left. \left\langle j^{-1} \nabla_{X_0} u^{-1}, (D.u_s) \circ u^{-1} \right\rangle_{H \rightarrow F} \right), j^{-1} h(X) \right\rangle_{H \rightarrow F}. \end{aligned}$$

(c) Let  $k \in H$  and consider the specific flow  $W^\rho := W + \rho \cdot jk$ ,  $\rho \in \mathbb{R}$ . We have,  $Q_\nu^{(m,r)}$ -a.e.

$$\begin{aligned} \langle D\psi \circ u, k \rangle_{H \rightarrow F} &= \frac{d^\pm}{d\rho} \Big|_{\rho=0} \psi \circ u(W^\rho) \\ &= \left\langle (D^u \psi) \circ u, j^{-1} \frac{d^\pm}{d\rho} \Big|_{\rho=0} u(W^\rho) \right\rangle_{H \rightarrow F} \\ &= \left\langle (D^u \psi) \circ u, j^{-1} \langle Du., k \rangle_{H \rightarrow F} \right\rangle_{H \rightarrow F} \\ &= \left\langle \left( \left\langle j^{-1} (Du.)_1, (D^u \psi) \circ u \right\rangle_{H \rightarrow F}, \left\langle j^{-1} \nabla_{W_0} u, (D^u \psi) \circ u \right\rangle_{H \rightarrow F} \right), k \right\rangle_{H \rightarrow F}. \end{aligned}$$

Furthermore, if  $k \in L^p(\Omega, Q_\nu^{(m,r)}; H)$  then

$$\begin{aligned} \|\langle D\psi \circ u, k \rangle_H\|_{L^1(\Omega, Q_\nu^{(m,r)})} &= \left\| \left\langle (D^u \psi) \circ u, j^{-1} \langle Du., k \rangle_{H \rightarrow F} \right\rangle_H \right\|_{L^1(\Omega, Q_\nu^{(m,r)})} \\ &\leq \|(D^u \psi) \circ u\|_{L^q(\Omega, Q_\nu^{(m,r)}; H)} \cdot \|j^{-1} \langle Du., k \rangle_{H \rightarrow F}\|_{L^p(\Omega, Q_\nu^{(m,r)}; H)} \\ &< \infty, \end{aligned}$$

cf. Definition 1.6 (iii). □

**Stochastic integral relative to  $W$ .** If  $u$  is, just for the next equality, the identity and  $\dot{f}(0, \cdot)(s) \in D_{p,1}$ ,  $s \in \mathcal{R}$ , and  $p$  is fixed in  $(W^\rho)_{\rho \in \mathbb{R}} \in \mathcal{F}_{p,1}(W)$  as a hypothesis of the present subsection, then  $Q_\nu^{(m,r)}$ -a.e.,

$$\begin{aligned} \delta \left( j^{-1} \dot{f}(0, \cdot) \right) (W) &= \sum_{i \in I(m,r)} \left\langle H_i, d\dot{f}(0, W) \right\rangle_{L^2} \cdot \langle H_i, dW \rangle_{L^2} \\ &\quad - \sum_{i \in I(m,r)} \left\langle (H_i, 0), D \left\langle H_i, d\dot{f}(0, W) \right\rangle_{L^2} \right\rangle_H \\ &\quad - \left\langle \dot{f}(0, W)(0), \frac{\nabla m(W_0)}{m(W_0)} \right\rangle_F - \left\langle \mathbf{e}, \nabla_{d, W_0} \dot{f}(0, W)(0) \right\rangle_F. \quad (2.12) \end{aligned}$$

Here  $\nabla_{W_0} \left\langle e_j, \dot{f}(0, W)(0) \right\rangle_F \equiv \nabla_{W_0} \left\langle e_j, \dot{f}(0, (\cdot, W_0))(0) \right\rangle_F$  is a gradient in the sense of Definition 1.1 (d) identifying in the argument  $W \equiv (W - W_0 \mathbf{1}, W_0)$ . To be precise, representation (2.12) follows, on the one hand, from  $(W^\rho)_{\rho \in \mathbb{R}} \in \mathcal{F}_{p,1}(W)$  which says  $j^{-1} \dot{f}(0, \cdot) \in L^p(\Omega, Q_\nu^{(m,r)}; H)$ . On the other hand, (2.12) is a consequence of an adaption of Theorem 6.7 (b) to the present situation. In particular recalling the meaning of  $Q$  relative to  $m$  in

Subsection 1.2, the stochastic integral (2.12) with integrator  $\dot{f}(0, \cdot)$  is well-defined if  $p > P$  where  $1/P + 1/Q = 1$ . Furthermore,

$$\left\langle H_i, d\dot{f}(0, W) \right\rangle_{L^2} \in D_{p,1} \equiv D_{p,1}(Q_\nu^{(m,r)})$$

by  $\dot{f}(0, \cdot)(s) \in D_{p,1}$ ,  $s \in \mathcal{R}$ , and the particular form of  $H_i$ .

**Substitution law and logarithmic derivative relative to  $X$ .** Being now interested in  $\beta_{h(X)}(X)$  where  $h$  is given by (2.10), the formula

$$\beta_h \circ u = -\delta \left( j^{-1} \dot{f}(0, \cdot) \right) \quad Q_\nu^{(m,r)}\text{-a.e.} \quad (2.13)$$

looks similar to the substitution law as, for example, demonstrated in [18], Theorem 2. Of course, we have to prove it. The proof is of instructive character. It provides some insight of how the Wiener space representation of the integral (2.13) is embedded in the general non-Gaussian stochastic calculus based on elementary integrals of the form  $-\beta_{j(H_i,0)}(X)$ . In fact, the representation of objects related to non-Gaussian stochastic calculus by means of Wiener space analysis terms is one of the major techniques in the paper.

Let  $Q$  be the number appearing in the definition of  $m$  in Subsection 1.2 and let  $q$  be given by condition (1') of Section 1. Recall Definitions 1.8 and 1.1 (b),(f), as well as Proposition 6.3.

Suppose that the operator  $(D^u, \mathcal{C}) := (j^* \circ \mathbb{D}^u, \mathcal{C}(\Omega^u))$  is closable on  $L^q(\Omega^u, P_\mu^{(m,r)}; H)$ . Assume  $1/p + 1/q = 1$  and let  $p' \geq q$  as well as  $1/p' + 1/q' = 1$  and note that this implies  $Q > p \geq q'$ . The following observation is important. If, for  $f \in H$ , there exists  $c_f > 0$  such that for all  $\psi \in D_{p',1}^u$  we have

$$\int \langle D^u \psi, f \rangle_H dP_\mu^{(m,r)} \leq c_f \|\psi\|_{L^{p'}(\Omega^u, P_\mu^{(m,r)})} \quad (2.14)$$

then there is a real element  $\beta_{jf} \in L^{q'}(\Omega^u, P_\mu^{(m,r)})$ , such that

$$-\int \psi \beta_{jf} dP_\mu^{(m,r)} = \int \langle D^u \psi, f \rangle_H dP_\mu^{(m,r)}. \quad (2.15)$$

**Proposition 2.5** *Let  $X = u(W)$  be a bijection  $C(\mathbb{R}; F) \rightarrow C(\mathbb{R}; F)$ . Assume that the conditions of Proposition 2.4 are satisfied. Suppose that the operator  $(D^u, \mathcal{C}) := (j^* \circ \mathbb{D}^u, \mathcal{C}(\Omega^u))$  is closable on  $L^q(\Omega^u, P_\mu^{(m,r)}; H)$ . Here  $q$  is given by condition (1') of Section 1.*

*Let  $m' \equiv m'(X - X_0 \mathbb{1}, \cdot)$  denote the conditional density of  $X_0$  with respect to  $\lambda_F$ . Assume that for given  $X - X_0 \mathbb{1}$ , the density  $m'$  satisfies the conditions on  $m$  in Subsection 1.2.*

*Let  $Q$  be the number appearing in the definition of  $m$  in Subsection 1.2 and let  $1/p + 1/q = 1$ . Suppose  $1/p + 1/Q < 1$ . Let moreover  $p' \in (1, \infty)$  and  $1/p' + 1/q' = 1$  such that  $p' < p$  and  $q' < Q$ .*

*Assume the following holds.*

(i) *We have (2.14) for  $f = (H_i, 0)$ ,  $i \in I(r)$ .*

(ii)  *$\dot{f}(0, \cdot)(s) \in D_{p,1}$ ,  $h(\cdot)(s) \in E_{p,1}^u$ ,  $s \in \mathcal{R}$ .*

$$(iii) \sum_{i \in I(r)} \left( \|\beta_{j(H_i, 0)}\|_{L^{q'}(\Omega^u, P_\mu^{(m, r)})} \vee 1 \right) \cdot \|\langle H_i, dh \rangle_{L^2}\|_{L^p(\Omega^u, P_\mu^{(m, r)})} < \infty.$$

In order to apply Lemma 2.3 we also shall suppose that

$$(iv) \text{ the linear span of } \{X_s : s \in \mathcal{R}\} \text{ is dense in } L^q(\Omega, Q_\nu^{(m, r)}; F).$$

Then we have (2.13),

$$\beta_h \circ u = -\delta(j^{-1}f(0, \cdot)) \quad Q_\nu^{(m, r)}\text{-a.e.},$$

and

$$\begin{aligned} \beta_{h(X)}(X) &= \sum_{i \in I(r)} (\langle H_i, dh(X) \rangle_{L^2} \cdot \beta_{j(H_i, 0)}(X) + \langle (H_i, 0), D^u \langle H_i, dh(X) \rangle_{L^2} \rangle_H) \\ &\quad + \left\langle h(X)(0), \frac{\nabla_{X_0} m'(X - X_0 \mathbb{I}, X_0)}{m'(X - X_0 \mathbb{I}, X_0)} \right\rangle_F + \langle e, \nabla_{d, X_0} h(X)(0) \rangle_F \end{aligned} \quad (2.16)$$

$P_\mu^{(m, r)}$ -a.e. where, with  $1/v = 1/p + 1/q'$ , the infinite sum  $\sum_{i \in I(r)}$  converges in  $L^v(\Omega^u, P_\mu^{(m, r)})$ . The term in the second line belongs to  $L^w(\Omega^u, P_\mu^{(m, r)})$  where  $1/w = 1/p + 1/Q$ .

**Remark.** (1) Summarizing all conditions on the exponents, we get first  $Q > 2$  from  $1/p + 1/q = 1$ ,  $1/p + 1/Q < 1$ , and (1'). With  $p' \geq q$  and  $Q > p \geq q'$  which has been noted for the definition in (2.14), (2.15) and  $p' < p$  from the formulation of Proposition 2.5 we get also  $p \geq q' > q$ ,  $p > p' \geq q$  and therefore  $Q > p > 2 > q > 1/(1 - 1/Q)$ .

Proof. *Step 1* First, we shall confirm well-definiteness of (2.16). Indeed by (i) and (2.15),  $\beta_{j(H_i, 0)} \in L^{q'}(\Omega^u, P_\mu^{(m, r)})$  is well-defined. Furthermore, by (ii) and the definition of  $H_i, i \in I(r)$ , we have  $\langle H_i, dh \rangle_{L^2} \in E_{p,1}^u \subset D_{p,1}^u$ . Now, the convergence of the infinite sum

$$\sum_{i \in I(r)} \langle H_i, dh(X) \rangle_{L^2} \cdot \beta_{j(H_i, 0)}(X)$$

in (2.16) in the norm of  $L^v(\Omega^u, P_\mu^{(m, r)})$  is a consequence of (iii). Furthermore, we mention that according to (ii) of the present proposition and Definition 1.10, (ii),

$$\sum_{i \in I(r)} D^u \langle H_i, dh(X) \rangle_{L^2} \cdot (H_i, 0)$$

is the coordinate representation of the projection of  $D^u j^{-1}h(X) \in L^p(\Omega^u, P_\mu^{(m, r)}; H \otimes H)$  to the linear span of  $\{(H_i, 0) : i \in I(r)\}$  and therefore an element of  $L^p(\Omega^u, P_\mu^{(m, r)}; H) \subseteq L^v(\Omega^u, P_\mu^{(m, r)}; H)$ .

Next, we shall verify that the right-hand side of (2.16) is indeed  $\beta_h$ , the logarithmic derivative of  $P_\mu^{(m, r)}$  in direction of  $h$ . Let  $\psi$  be a cylindrical function on  $\Omega^u$  of the form  $\psi(X) = g_0(X_0) \cdot g_1(X_{t_1} - X_0, \dots, X_{t_k} - X_0)$  where  $g_0 \in C_b^1(F)$ ,  $g_1 \in C_b^1(F^k)$ , and  $t_l \in \mathcal{R} \cap \{z \cdot t/2^m : z \in \mathbb{Z}\}$ ,  $l \in \{1, \dots, k\}$ . Similar to (6.5) and (6.6),

$$\begin{aligned} \langle (D^u \psi)(X), j^{-1} \rho \rangle_H &= g_0(X_0) \cdot \sum_{i=1}^k \langle \nabla_i g_1(X_{t_1} - X_0, \dots, X_{t_k} - X_0), \rho_{t_i} - \rho_0 \rangle_F \\ &\quad + \langle \nabla g_0(X_0), \rho_0 \rangle_F \cdot g_1(X_{t_1} - X_0, \dots, X_{t_k} - X_0) \end{aligned} \quad (2.17)$$

with  $\rho \equiv (\rho - \rho_0 \cdot \mathbb{1}, \rho_0) \in \{j(f, x) : (f, x) \in H\}$  and  $\nabla_i$  denoting the gradient with respect to the  $i$ -th entry. By Proposition 2.4 (a),  $h$  is  $P_\mu^{(m,r)}$ -a.e. absolutely continuous and  $j^{-1}h \in L^p(\Omega^u, P_\mu^{(m,r)}; H)$ . We have

$$\langle D^u \psi, j^{-1}h \rangle_H = \sum_{i \in I(r)} \langle D^u \psi, (H_i, 0) \rangle_H \langle H_i, dh \rangle_{L^2} + \langle \nabla_{X_0} \psi, h(0) \rangle_F$$

where the convergence of the right-hand side in  $L^p(\Omega^u, P_\mu^{(m,r)})$  can be deduced from (iii) and (2.17). Therefore,

$$\begin{aligned} & \int \langle D^u \psi, j^{-1}h \rangle_H dP_\mu^{(m,r)} \\ &= \sum_{i \in I(r)} \int \langle D^u \psi, (H_i, 0) \rangle_H \langle H_i, dh \rangle_{L^2} dP_\mu^{(m,r)} + \int \langle \nabla_{X_0} \psi, h(0) \rangle_F dP_\mu^{(m,r)} \\ &= - \sum_{i \in I(r)} \int \psi \cdot \langle (H_i, 0), D^u \langle H_i, dh \rangle_{L^2} \rangle_H dP_\mu^{(m,r)} - \int \psi \cdot \langle \mathbf{e}, \nabla_{d, X_0} h(0) \rangle_F dP_\mu^{(m,r)} \\ &\quad - \sum_{i \in I(r)} \int \psi \beta_{j(H_i, 0)} \cdot \langle H_i, dh \rangle_{L^2} dP_\mu^{(m,r)} - \int \psi \cdot \left\langle h(0), \frac{\nabla_{X_0} m'}{m'} \right\rangle_F dP_\mu^{(m,r)} \end{aligned}$$

which first yields the existence of the logarithmic derivative of  $P_\mu^{(m,r)}$  in direction of  $h$ ,  $\beta_h \in L^v(\Omega^u, P_\mu^{(m,r)})$  (since  $v < w$ ), and then (2.16). In addition, we have (2.14) for  $f = (0, e_i)$ ,  $i \in \{1, \dots, n \cdot d\}$  with

$$\beta_{j(0, e_i)}(X) = \frac{(\nabla_{X_0} m')_i (X - X_0 \mathbb{1}, X_0)}{m'(X - X_0 \mathbb{1}, X_0)} \quad P_\mu^{(m,r)}\text{-a.e.}$$

It is the objective of the remaining proof to point out that relation (2.13) is a consequence of (2.12) and (2.16). Strictly speaking, we have to demonstrate that,  $Q_\nu^{(m,r)}$ -a.e.,

$$\begin{aligned} & - \sum_{i \in I(r)} \left( \langle H_i, dh \circ u \rangle_{L^2} \cdot \beta_{j(H_i, 0)} \circ u + \langle (H_i, 0), (D^u \langle H_i, dh \rangle_{L^2}) \circ u \rangle_H \right) \\ & \quad - \sum_i (h \circ u(0))_i \cdot \beta_{j(0, e_i)} \circ u - \langle \mathbf{e}, \nabla_{d, X_0} h \circ u(0) \rangle_H \\ &= \sum_{j \in I(m, r)} \langle H_j, d\dot{f}(0, \cdot) \rangle_{L^2} \cdot \langle H_j, dW \rangle_{L^2} - \sum_{j \in I(m, r)} \left\langle (H_j, 0), D \langle H_j, d\dot{f}(0, \cdot) \rangle_{L^2} \right\rangle_H \\ & \quad - \left\langle \dot{f}(0, \cdot)(0), \frac{\nabla m(W_0)}{m(W_0)} \right\rangle_F - \left\langle \mathbf{e}, \nabla_{d, W_0} \dot{f}(0, \cdot)(0) \right\rangle_F. \end{aligned} \tag{2.18}$$

We also recall (2.9) which leads to

$$\left\langle Du_0, j^{-1} \dot{f}(0, \cdot) \right\rangle_{H \rightarrow F} = h \circ u(0). \tag{2.19}$$

*Step 2* In this step, we are going to analyze the terms  $\langle H_i, dh \circ u \rangle_{L^2}$  and  $(D^u \langle H_i, dh \rangle_{L^2}) \circ u$  from the left-hand side of (2.18). We stress that everything further in this step will hold almost everywhere under the measure  $Q_\nu^{(m,r)}$ .

Recalling,  $u^{-1} \in K_{q,1}^u$ , (cf. (ii) of Proposition 2.4) we may define

$$\begin{aligned}\mathbf{H}_i &:= q_{m,r} j^{-1} \langle D^u u^{-1} \circ u, (H_i, 0) \rangle_{H \rightarrow F}, \quad i \in I(r), \\ \mathbf{e}_j &:= q_{m,r} j^{-1} \langle D^u u^{-1} \circ u, (0, e_j) \rangle_{H \rightarrow F}, \quad j \in \{1, \dots, n \cdot d\}.\end{aligned}\tag{2.20}$$

In fact, we have

$$\mathbf{H}_i, \mathbf{e}_j \in H^{(m,r)} \quad Q_\nu^{(m,r)} \text{ a.e.}$$

and with Definition 1.10 (iii)

$$\mathbf{H}_i, \mathbf{e}_j \in L^p(\Omega^u, P_\mu^{(m,r)}; H^{(m,r)}), \quad i \in I(r), \quad j \in \{1, \dots, n \cdot d\}.$$

Moreover, Lemma 2.2 implies

$$\langle D.u, q_{m,r} j^{-1} \langle D^u u^{-1} \circ u, (H_i, 0) \rangle_{H \rightarrow F} \rangle_{H \rightarrow F} = \langle D.u, j^{-1} \langle D^u u^{-1} \circ u, (H_i, 0) \rangle_{H \rightarrow F} \rangle_{H \rightarrow F}$$

where, as in the proof of Proposition 2.4, the dots indicate the variable to be jointly integrated over. Thus, Proposition 2.4 (b) as well as the arguments of its proof together with (2.20) yield

$$\begin{aligned}(j(H_i, 0))_s &= \left\langle \left( \langle j^{-1} (D^u u^{-1})_1 \circ u, D.u_s \rangle_{H \rightarrow F}, \langle j^{-1} \nabla_{X_0} u^{-1} \circ u, D.u_s \rangle_{H \rightarrow F} \right), (H_i, 0) \right\rangle_{H \rightarrow F} \\ &= \langle D.u_s, \langle j^{-1} D^u u^{-1} \circ u, (H_i, 0) \rangle_{H \rightarrow F} \rangle_{H \rightarrow F} \\ &= \langle D.u_s, j^{-1} \langle D^u u^{-1} \circ u, (H_i, 0) \rangle_{H \rightarrow F} \rangle_{H \rightarrow F} \\ &= \langle D.u_s, \mathbf{H}_i \rangle_{H \rightarrow F}, \quad s \in \mathcal{R}, \quad i \in I(r).\end{aligned}\tag{2.21}$$

Similarly,

$$(0, e_j) = \langle D.u_s, \mathbf{e}_j \rangle_{H \rightarrow F}, \quad s \in \mathcal{R}, \quad j \in \{1, \dots, n \cdot d\}.\tag{2.22}$$

As a consequence of Proposition 2.4 (a),  $(X^\rho)_{\rho \in \mathbb{R}} \in \mathcal{F}_{p,1}(X)$ , we have  $h \circ u \in jH \quad Q_\nu^{(m,r)}$ -a.e. Also, we have  $Q_\nu^{(m,r)}$ -a.e. for all  $s \in \mathcal{R}$

$$\begin{aligned}\langle j^{-1} D^u u_s^{-1} \circ u, j^{-1} h \circ u \rangle_H &= \sum_{i \in I(r)} \langle j^{-1} D^u u_s^{-1} \circ u, (H_i, 0) \rangle_H \cdot \langle (H_i, 0), j^{-1} h \circ u \rangle_H \\ &\quad + \langle (D^u u_s^{-1} \circ u)_2, h \circ u(0) \rangle_F.\end{aligned}$$

Thus, the sum

$$\sum_{i \in I(r)} \mathbf{H}_i \cdot \langle (H_i, 0), j^{-1} h \circ u \rangle_H \quad \text{converges } Q_\nu^{(m,r)}\text{-a.e. in } H.$$

By (2.19)-(2.22) we get therefore

$$\begin{aligned}
& \left\langle Du, \sum_{i \in I(r)} \mathbf{H}_i(\cdot) \cdot \langle H_i, dh \circ u \rangle_{L^2} + \sum_j \mathbf{e}_j(\cdot) \cdot \langle e_j, h \circ u(0) \rangle_F \right\rangle_{H \rightarrow F} \\
&= \sum_{i \in I(r)} \langle Du, \mathbf{H}_i(\cdot) \rangle_{H \rightarrow F} \cdot \langle H_i, dh \circ u \rangle_{L^2} + \sum_j \langle Du, \mathbf{e}_j(\cdot) \rangle_{H \rightarrow F} \cdot \langle e_j, h \circ u(0) \rangle_F \\
&= \sum_{i \in I(r)} j(H_i, 0) \langle (H_i, 0), j^{-1}h \circ u \rangle_H + h \circ u(0) \cdot \mathbb{1} \\
&= h \circ u = \left\langle Du, j^{-1}\dot{f}(0, \cdot) \right\rangle_{H \rightarrow F},
\end{aligned}$$

for the second last equality sign we consider the integral induced by the embedding  $j$  the inner product in  $H$  with  $\mathbb{1} \in H$ . Using Lemma 2.3 it follows that

$$\begin{aligned}
& \sum_{i \in I(r)} \langle (H_j, 0), \mathbf{H}_i \rangle_H \cdot \langle H_i, dh \circ u \rangle_{L^2} + \sum_i \langle (H_j, 0), \mathbf{e}_i \rangle_H \cdot (h \circ u(0))_i = \langle H_j, d\dot{f}(0, \cdot) \rangle_{L^2}, \\
& \sum_{j \in I(r)} \langle (0, e_i), \mathbf{H}_j \rangle_H \cdot \langle H_j, dh \circ u \rangle_{L^2} + \sum_j \langle (0, e_i), \mathbf{e}_j \rangle_H \cdot (h \circ u(0))_j = \langle e_i, \dot{f}(0, \cdot)(0) \rangle_F,
\end{aligned} \tag{2.23}$$

$j \in I(m, r)$ ,  $i \in \{1, \dots, n \cdot d\}$ . Furthermore, we mention that  $\langle H_i, dh \rangle_{L^2} \in E_{p,1}^u$  by (ii) and the special shape of  $H_i$ ,  $i \in I(r)$ . Now, by Proposition 2.4 (c), the arguments of the proof of 2.4 (b), and (2.21) for the last equality sign give

$$\begin{aligned}
& \sum_{j \in I(m,r)} \langle \mathbf{H}_i, (H_j, 0) \rangle_H \langle (H_j, 0), D\langle H_i, dh \circ u \rangle_{L^2} \rangle_H \\
& \quad + \sum_j \langle \mathbf{H}_i, (0, e_j) \rangle_H \langle (0, e_j), D\langle H_i, dh \circ u \rangle_{L^2} \rangle_H \\
&= \langle \mathbf{H}_i, D\langle H_i, dh \circ u \rangle_{L^2} \rangle_H \\
&= \langle \mathbf{H}_i, \langle j^{-1}Du, (D^u\langle H_i, dh \rangle_{L^2}) \circ u \rangle_{H \rightarrow F} \rangle_H \\
&= \langle \langle j^{-1}Du, \mathbf{H}_i \rangle_{H \rightarrow F}, (D^u\langle H_i, dh \rangle_{L^2}) \circ u \rangle_H \\
&= \langle j^{-1}\langle Du, \mathbf{H}_i \rangle_{H \rightarrow F}, (D^u\langle H_i, dh \rangle_{L^2}) \circ u \rangle_H \\
&= \langle (H_i, 0), (D^u\langle H_i, dh \rangle_{L^2}) \circ u \rangle_H, \quad i \in I(r).
\end{aligned} \tag{2.24}$$

Similarly, with (2.22) for the last equality sign we obtain

$$\begin{aligned}
& \sum_i \sum_{j \in I(m,r)} \langle \mathbf{e}_i, (H_j, 0) \rangle_H \langle (H_j, 0), D\langle e_i, h \circ u(0) \rangle_F \rangle_H \\
& \quad + \sum_i \sum_j \langle \mathbf{e}_i, (0, e_j) \rangle_H \langle (0, e_j), D\langle e_i, h \circ u(0) \rangle_F \rangle_H \\
&= \sum_i \langle \mathbf{e}_i, D\langle e_i, h \circ u(0) \rangle_F \rangle_H
\end{aligned}$$

$$\begin{aligned}
&= \sum_i \langle \mathbf{e}_i, \langle j^{-1} Du., D^u \langle e_i, h \circ u(0) \rangle_F \rangle_{H \rightarrow F} \rangle_H \\
&= \sum_i \langle \langle j^{-1} Du., \mathbf{e}_i \rangle_{H \rightarrow F}, D^u \langle e_i, h \circ u(0) \rangle_F \rangle_H \\
&= \sum_i \langle j^{-1} \langle Du., \mathbf{e}_i \rangle_{H \rightarrow F}, D^u \langle e_i, h \circ u(0) \rangle_F \rangle_H \\
&= \langle \mathbf{e}, \nabla_{d, X_0} h \circ u(0) \rangle_H.
\end{aligned} \tag{2.25}$$

*Step 3* We are interested in the logarithmic derivatives  $\beta_{j(H_i, 0)}$  and  $\beta_{j(0, e_{j'})}$ . We recall that by (2.20) and  $u^{-1} \in K_{q,1}^u$  we have  $\mathbf{H}_i \in L^p(\Omega, Q_\nu^{(m,r)}; H)$ ,  $i \in I(r)$ , cf. Definition 1.10, (iii). Let  $\psi \in \mathcal{C}(\Omega^u)$  be a cylindric cylindrical function in the sense of Definitions 1.1 (a) and 1.8. By Remark (9) of Section 1 it holds that  $\psi \in E_{q,1}^u$ .

According to assumption (i) of this proposition, (2.21), (2.22),  $u \in K_{q,1}$ , and Proposition 2.4 (c) we have

$$\begin{aligned}
- \int \psi \cdot \beta_{j(H_i, 0)} dP_\mu^{(m,r)} &= \int \langle D^u \psi, (H_i, 0) \rangle_H dP_\mu^{(m,r)} \\
&= \int \langle (D^u \psi) \circ u, j^{-1} \langle Du., \mathbf{H}_i \rangle_{H \rightarrow F} \rangle_H dQ_\nu^{(m,r)} \\
&= \int \langle (D^u \psi) \circ u, \langle j^{-1} Du., \mathbf{H}_i \rangle_{H \rightarrow F} \rangle_H dQ_\nu^{(m,r)} \\
&= \int \langle \langle (D^u \psi) \circ u, j^{-1} Du. \rangle_{H \rightarrow F}, \mathbf{H}_i \rangle_H dQ_\nu^{(m,r)} \\
&= \int \langle D\psi \circ u, \mathbf{H}_i \rangle_H dQ_\nu^{(m,r)} \\
&= \int \psi \circ u \cdot \delta(\mathbf{H}_i) dQ_\nu^{(m,r)}
\end{aligned}$$

and similarly

$$- \int \psi \cdot \beta_{j(0, e_{j'})} dP_\mu^{(m,r)} = \int \psi \circ u \cdot \delta(\mathbf{e}_{j'}) dQ_\nu^{(m,r)}.$$

This shows

$$\beta_{j(H_i, 0)} \circ u = -\delta(\mathbf{H}_i) \quad Q_\nu^{(m,r)}\text{-a.e.}, \quad i \in I(r),$$

and

$$\beta_{j(0, e_{j'})} \circ u = -\delta(\mathbf{e}_{j'}) \quad Q_\nu^{(m,r)}\text{-a.e.}, \quad j' \in \{1, \dots, n \cdot d\}.$$

Taking into consideration  $\mathbf{H}_i \in H^{(m,r)}$ ,  $i \in I(r)$ , as well as  $\mathbf{e}_j \in H^{(m,r)}$ ,  $j \in \{1, \dots, n \cdot d\}$   $Q_\nu^{(m,r)}$ -a.e., we get  $\mathbf{H}_i = \sum_{j \in I(m,r)} \langle \mathbf{H}_i, (H_j, 0) \rangle_H \cdot (H_j, 0) + \sum_{j'} \langle \mathbf{H}_i, (0, e_{j'}) \rangle_H \cdot (0, e_{j'})$  and thus from Theorem 6.7 (b) the representation

$$\beta_{j(H_i, 0)} \circ u = -\delta(\mathbf{H}_i)$$

$$\begin{aligned}
&= - \sum_{j \in I(m,r)} \langle \mathbf{H}_i, (H_j, 0) \rangle_H \cdot \langle H_j, dW \rangle_{L^2} \\
&\quad + \sum_{j \in I(m,r)} \langle (H_j, 0), D \langle \mathbf{H}_i, (H_j, 0) \rangle_H \rangle_H \\
&\quad + \left\langle (\mathbf{H}_i)_2, \frac{\nabla m(W_0)}{m(W_0)} \right\rangle_F + \langle \mathbf{e}, \nabla_{d,W_0}(\mathbf{H}_i)_2 \rangle_F, \quad i \in I(r). \quad (2.26)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\beta_{j(0,e_i)} \circ u &= -\delta(\mathbf{e}_i) \\
&= - \sum_{j \in I(m,r)} \langle \mathbf{e}_i, (H_j, 0) \rangle_H \cdot \langle H_j, dW \rangle_{L^2} \\
&\quad + \sum_{j \in I(m,r)} \langle (H_j, 0), D \langle \mathbf{e}_i, (H_j, 0) \rangle_H \rangle_H \\
&\quad + \left\langle (\mathbf{e}_i)_2, \frac{\nabla m(W_0)}{m(W_0)} \right\rangle_F + \langle \mathbf{e}, \nabla_{d,W_0}(\mathbf{e}_i)_2 \rangle_F, \quad i \in \{1, \dots, n \cdot d\}. \quad (2.27)
\end{aligned}$$

*Step 4* We will finally show that (2.13) is a consequence of (2.16). For this we verify (2.18) by using the results of the Steps 2 and 3. Especially (2.24)-(2.27) imply that we have  $Q_\nu^{(m,r)}$ -a.e.

$$\begin{aligned}
&- \sum_{i \in I(r)} \left( \langle H_i, dh \circ u \rangle_{L^2} \cdot \beta_{j(H_i,0)} \circ u + \langle (H_i, 0), (D^u \langle H_i, dh \rangle_{L^2}) \circ u \rangle_H \right) \\
&\quad - \sum_i (h \circ u(0))_i \cdot \beta_{j(0,e_i)} \circ u - \langle \mathbf{e}, \nabla_{d,X_0} h \circ u(0) \rangle_H \\
&= \sum_{i \in I(r)} \langle H_i, dh \circ u \rangle_{L^2} \cdot \sum_{j \in I(m,r)} \langle \mathbf{H}_i, (H_j, 0) \rangle_H \cdot \langle H_j, dW \rangle_{L^2} \\
&\quad - \sum_{i \in I(r)} \langle H_i, dh \circ u \rangle_{L^2} \cdot \sum_{j \in I(m,r)} \langle (H_j, 0), D \langle \mathbf{H}_i, (H_j, 0) \rangle_H \rangle_H \\
&\quad - \sum_{i \in I(r)} \langle H_i, dh \circ u \rangle_{L^2} \cdot \left( \left\langle (\mathbf{H}_i)_2, \frac{\nabla m(W_0)}{m(W_0)} \right\rangle_F + \langle \mathbf{e}, \nabla_{d,W_0}(\mathbf{H}_i)_2 \rangle_F \right) \\
&\quad - \sum_{i \in I(r)} \sum_{j \in I(m,r)} \langle \mathbf{H}_i, (H_j, 0) \rangle_H \langle (H_j, 0), D \langle H_i, dh \circ u \rangle_{L^2} \rangle_H \\
&\quad - \sum_{i \in I(r)} \sum_{j'} \langle \mathbf{H}_i, (0, e_{j'}) \rangle_H \langle (0, e_{j'}), D \langle H_i, dh \circ u \rangle_{L^2} \rangle_H \\
&\quad + \sum_i (h \circ u(0))_i \cdot \sum_{j \in I(m,r)} \langle \mathbf{e}_i, (H_j, 0) \rangle_H \cdot \langle H_j, dW \rangle_{L^2} \\
&\quad - \sum_i (h \circ u(0))_i \cdot \sum_{j \in I(m,r)} \langle (H_j, 0), D \langle \mathbf{e}_i, (H_j, 0) \rangle_H \rangle_H \\
&\quad - \sum_i (h \circ u(0))_i \cdot \left( \left\langle (\mathbf{e}_i)_2, \frac{\nabla m(W_0)}{m(W_0)} \right\rangle_F + \langle \mathbf{e}, \nabla_{d,W_0}(\mathbf{e}_i)_2 \rangle_F \right) \\
&\quad - \sum_i \sum_{j \in I(m,r)} \langle \mathbf{e}_i, (H_j, 0) \rangle_H \langle (H_j, 0), D \langle h \circ u(0) \rangle_i \rangle_H
\end{aligned}$$

$$- \sum_i \sum_{j'} \langle \mathbf{e}_i, (0, e_{j'}) \rangle_H \langle (0, e_{j'}), D(h \circ u(0))_i \rangle_H .$$

Applying the usual product rule, this is equal to

$$\begin{aligned} & \sum_{i \in I(r)} \langle H_i, dh \circ u \rangle_{L^2} \cdot \sum_{j \in I(m, r)} \langle \mathbf{H}_i, (H_j, 0) \rangle_H \cdot \langle H_j, dW \rangle_{L^2} \\ & + \sum_i (h \circ u(0))_i \cdot \sum_{j \in I(m, r)} \langle \mathbf{e}_i, (H_j, 0) \rangle_H \cdot \langle H_j, dW \rangle_{L^2} \\ & - \sum_{i \in I(r)} \sum_{j \in I(m, r)} \left\langle (H_j, 0), D \left( \langle \mathbf{H}_i, (H_j, 0) \rangle_H \cdot \langle H_i, dh \circ u \rangle_{L^2} \right) \right\rangle_H \\ & - \sum_i \sum_{j \in I(m, r)} \left\langle (H_j, 0), D \left( \langle \mathbf{e}_i, (H_j, 0) \rangle_H \cdot (h \circ u(0))_i \right) \right\rangle_H \\ & - \sum_{i \in I(r)} \sum_{j'} \langle \mathbf{H}_i, (0, e_{j'}) \rangle_H \langle (0, e_{j'}), D \langle H_i, dh \circ u \rangle_{L^2} \rangle_H \\ & - \sum_i \sum_{j'} \langle \mathbf{e}_i, (0, e_{j'}) \rangle_H \langle (0, e_{j'}), D(h \circ u(0))_i \rangle_H \\ & - \sum_{i \in I(r)} \sum_{j'} \langle H_i, dh \circ u \rangle_{L^2} \cdot \left( \left\langle \langle \mathbf{H}_i, (0, e_{j'}) \rangle_H \cdot e_{j'}, \frac{\nabla m(W_0)}{m(W_0)} \right\rangle_F \right. \\ & \quad \left. + \langle D \langle \mathbf{H}_i, (0, e_{j'}) \rangle_H, (0, e_{j'}) \rangle_H \right) \\ & - \sum_i \sum_{j'} (h \circ u(0))_i \cdot \left( \left\langle \langle \mathbf{e}_i, (0, e_{j'}) \rangle_H \cdot e_{j'}, \frac{\nabla m(W_0)}{m(W_0)} \right\rangle_F \right. \\ & \quad \left. + \langle D \langle \mathbf{e}_i, (0, e_{j'}) \rangle_H, (0, e_{j'}) \rangle_H \right) . \end{aligned}$$

Now (2.23) yields

$$\begin{aligned} & - \sum_{i \in I(r)} \left( \langle H_i, dh \circ u \rangle_{L^2} \cdot \beta_{j(H_i, 0)} \circ u + \langle (H_i, 0), (D^u \langle H_i, dh \rangle_{L^2}) \circ u \rangle_H \right) \\ & - \sum_i (h \circ u(0))_i \cdot \beta_{j(0, e_i)} \circ u - \langle \mathbf{e}, \nabla_{d, X_0} h \circ u(0) \rangle_H \\ & = \sum_{j \in I(m, r)} \langle H_j, d\dot{f}(0, \cdot) \rangle_{L^2} \cdot \langle H_j, dW \rangle_{L^2} - \sum_{j \in I(m, r)} \left\langle (H_j, 0), D \langle H_j, d\dot{f}(0, \cdot) \rangle_{L^2} \right\rangle_H \\ & - \left\langle \dot{f}(0, \cdot)(0), \frac{\nabla m(W_0)}{m(W_0)} \right\rangle_F - \left\langle \mathbf{e}, \nabla_{d, W_0} \dot{f}(0, \cdot)(0) \right\rangle_F \end{aligned}$$

which is (2.18).  $\square$

### 2.3 A chain rule for absolutely continuous $f(\rho, W)(\cdot)$ but jumps in $\dot{f}(\rho, W)(\cdot)$

We proceed with a definition whose appropriateness will become obvious by the subsequent Lemma 3.1. For  $m \in \mathbb{Z}_+$  and  $r \in \mathbb{N}$  introduce the following.

**Definition 2.6** Let  $1 < p < \infty$  and let  $Y \in I^p$  such that  $Y_s$  is constant outside of  $s \in \mathcal{R}$ . Let  $f = f_c + f_j$  where  $f_c \in L^q(\Omega, Q_\nu^{(m,r)}; L^2)$  is a  $Q_\nu^{(m,r)}$ -a.e. continuous process and  $f_j$  is a cadlag pure jump process of finite jump variation on  $\mathcal{R}$ ,  $V_{-rt}^{(r-1)t}(f) \in L^q(\Omega, Q_\nu^{(m,r)})$ ,  $1/p + 1/q = 1$ . Let  $x \in L^q(\Omega, Q_\nu^{(m,r)}; F)$ .

(a) Let  $\varphi = (f, x)$  and define

$$\begin{aligned} \int_u^v \langle f, dY \rangle_F + \langle x, \tfrac{1}{2}(Y_{0-} + Y_0) \rangle_F &:= \langle (f \mathbb{1}_{[u,v]}, x), j^{-1}Y_a \rangle_H \\ &+ \sum_{w \in (u,v)} \frac{1}{2} \langle f_{w-} + f_w, \Delta Y_j(w) \rangle_F + \frac{1}{2} \langle f_u, \Delta Y_j(u) \rangle_F + \frac{1}{2} \langle f_{v-}, \Delta Y_j(v) \rangle_F \quad Q_\nu^{(m,r)}\text{-a.e.} \end{aligned}$$

noting that  $\langle (f \mathbb{1}_{[u,v]}, x), j^{-1}Y_a \rangle_H = \int_u^v \langle f, dY_a \rangle_F + \langle x, \tfrac{1}{2}(Y_{0-} + Y_0) \rangle_F$  in the sense of Riemann-Stieltjes integration.

(b) Let  $\varphi = (f, x)$  and define  $j^{-1}Y$  applied to the test element  $\varphi$  by

$$j^{-1}Y(\varphi) := \int_{\mathcal{R}} \langle f, dY \rangle_F + \langle x, \tfrac{1}{2}(Y_{0-} + Y_0) \rangle_F \quad Q_\nu^{(m,r)}\text{-a.e.} \quad (2.28)$$

(c) For  $\varphi$  and  $Y$  as above, we introduce the inner product  $\langle \cdot, j^{-1}(\cdot) \rangle_H$  by

$$\langle \varphi, j^{-1}Y \rangle_H := j^{-1}Y(\varphi).$$

Replacing in (2.28)  $\langle \cdot, \cdot \rangle_F$  with  $\langle \cdot, \cdot \rangle_{F \rightarrow F}$ , we define  $\langle \varphi, j^{-1}Y \rangle_{H \rightarrow F}$ .

**Relations (2.9) and (2.10) in a distributional sense.** Suppose we are given  $W^\rho := W + f(\rho, W)$ . Furthermore, allow

$$\dot{f}(0, \cdot) \in I^p \quad (2.29)$$

where  $1/p + 1/q = 1$  and  $q$  is fixed by assumption (1') of Section 1 and let (2.6) and (2.7) be in force. In particular, we recall that the symbol  $\dot{f}(\rho, W)$  is used in the strict sense of a *weak mixed derivative* which is  $Q_\nu^{(m,r)}$ -a.e. given for all test elements  $k = (g, x) \in H$  with cadlag  $g : \mathcal{R} \rightarrow F$  of finite jump variation,  $x \in F$ , by (2.7),

$$\begin{aligned} \dot{f}(\rho, W)(k) &= \frac{d^\pm}{d\rho} \langle g, df^0(\rho, W) \rangle_{L^2} + \langle x, \dot{B}_\rho \rangle_F \\ &= \frac{1}{2} \left( \frac{d^-}{d\rho} \langle g, df^0(\rho, W) \rangle_{L^2} + \frac{d^+}{d\rho} \langle g, df^0(\rho, W) \rangle_{L^2} \right) + \langle x, \dot{B}_\rho \rangle_F. \end{aligned}$$

One major difference to the analysis in Subsection 2.2 is that we assume there  $(W^\rho)_{\rho \in \mathbb{R}} \in \mathcal{F}_{p,1}$  which is replaced here by (2.29) and (2.7) for  $\rho = 0$ . Even more restrictive in Subsection 2.2 is the hypothesis  $u_s \in E_{q,1}$ ,  $s \in \mathcal{R}$ , as part of  $u \in K_{q,1}$  in order to establish the generator of the flow  $(X^\rho)_{\rho \in \mathbb{R}}$  namely relation (2.10). Note also that for the following proposition we do not require that  $W^\rho = W + f(\rho, W)$ ,  $\rho \in \mathbb{R}$ , is a flow.

As a preparation of the analysis in Section 3, we are interested in a counterpart of (2.10) at  $\rho = 0$  under the conditions of the present subsection.

For this fix  $r \in \mathbb{Z}_+$  and define for all  $W \in jH$  the terms  $\xi_i := \langle (H_i, 0), j^{-1}W \rangle_H$ ,  $i \in I(r)$ ,  $x_j := (W_0)_j$ ,  $j \in \{1, \dots, n \cdot d\}$ , and

$$Z_s(X; W) \equiv Z_s(X; \dots, \xi_i, \dots; \dots, x_j, \dots) := X_s \left( \sum_{i \in I(r)} \xi_i \cdot j(H_i, 0) + x \mathbb{I} \right).$$

**Proposition 2.7** *Assume (1') of Section 1. Let  $1 < q < \infty$  be the number defined in (1') and let  $1/p + 1/q = 1$ . Assume the following.*

(i) *We have (2.29),  $\dot{f}(0, \cdot) \in I^p$ .*

(ii) *For  $W^\rho = W + f(\rho, W)$ ,  $\rho \in \mathbb{R}$ , we have (2.7) at  $\rho = 0$ , that is  $Q_\nu^{(m,r)}$ -a.e. there exists the weak mixed derivative*

$$\dot{f}(0, W)(k) = \left. \frac{d^\pm}{d\rho} \right|_{\rho=0} \langle g, df^0(\rho, W) \rangle_{L^2} + \langle x, \dot{B}_0 \rangle_F.$$

*Furthermore, for all test elements  $k = (g, x) \in H$  with  $g = g_c + g_j$  where  $g_c \in C(\mathcal{R}; F)$  and  $g_j : \mathcal{R} \rightarrow F$  being a cadlag pure jump function of finite jump variation and  $x \in F$  we have the representation*

$$\dot{f}(0, \cdot)(k) = \left\langle k, j^{-1} \dot{f}(0, \cdot) \right\rangle_H,$$

*the right-hand side in the sense of Definition 2.6 (c).*

(iii)  *$Q_\nu^{(m,r)}$ -a.e.,  $Z_s(X; \dots, \xi_i, \dots; \dots, x_j, \dots) : l^2 \rightarrow F$  is two times Fréchet differentiable for all  $s \in \mathcal{R}$ . This implies for  $Q_\nu^{(m,r)}$ -a.e.  $W \in \Omega$  the existence of the Fréchet derivative  $D_F X_s(W)$  in the sense of Subsection 1.2 (after condition (5)) for which we assume that*

$$D_{F,1} X_s(W) = (D_{F,1} X_s(W))_c + (D_{F,1} X_s(W))_j \quad (2.30)$$

*where  $(D_{F,1} X_s(W))_c \in C(\mathcal{R}; F)$  and  $(D_{F,1} X_s(W))_j : \mathcal{R} \rightarrow F$  is a cadlag pure jump function of finite jump variation.*

(iv) *For  $Q_\nu^{(m,r)}$ -a.e.  $W \in \Omega$  there is some  $0 < c \equiv c(W) < \infty$  such that*

$$\langle h, j^{-1}(W^\rho - W) \rangle_H = \langle h, j^{-1}f(\rho, W) \rangle_H \leq c \|h\|_H \cdot |\rho|, \quad h \in H,$$

*as well as*

$$\|j^{-1}(W^\rho - W)\|_H^2 = \|j^{-1}f(\rho, W)\|_H^2 \leq c \cdot |\rho|, \quad |\rho| < 1.$$

*Then  $\left. \frac{d^\pm}{d\rho} \right|_{\rho=0} X_s(W^\rho)$  exists for  $Q_\nu^{(m,r)}$ -a.e.  $W \in \Omega$  and all  $s \in \mathcal{R}$  and we have*

$$\begin{aligned} h \circ u(W)(s) &= \left. \frac{d^\pm}{d\rho} \right|_{\rho=0} X_s(W^\rho) \\ &= \left. \frac{d^\pm}{d\rho} \right|_{\rho=0} \langle D_F X_s(W), j^{-1}f(\rho, W) \rangle_H = \left\langle D_F X_s(W), j^{-1} \dot{f}(0, W) \right\rangle_{H \rightarrow F}. \end{aligned}$$

*Furthermore, this point wise calculated expression belongs to  $L^1(\Omega, Q_\nu^{(m,r)}; F)$  whenever  $\nabla_H X_s \in L^q(\Omega, Q_\nu^{(m,r)}; H)$ .*

**Remark. (2)** The existence of the second order partial directional derivatives of  $X_s$  with respect to  $\kappa_i$ ,  $i \in I(r)$ , and all  $\lambda_j$  such that

$$\sum_{i,i' \in I(r)} \left( \frac{\partial^2 X_s}{\partial \kappa_i \partial \kappa_{i'}} \right)^2 + \sum_{i \in I(r), j'} \left( \frac{\partial^2 X_s}{\partial \kappa_i \partial \lambda_{j'}} \right)^2 < \infty$$

together with the continuity of

$$\begin{aligned} D_F^2 X_s &\equiv \sum_{i,i' \in I(r)} \frac{\partial^2 X_s}{\partial \kappa_i \partial \kappa_{i'}} (H_i, 0) \times (H_{i'}, 0) + \sum_{j,j'} \frac{\partial^2 X_s}{\partial \lambda_j \partial \lambda_{j'}} (0, e_{j'}) \times (0, e_{j'}) \\ &+ 2 \sum_{i \in I(r), j'} \frac{\partial^2 X_s}{\partial \kappa_i \partial \lambda_{j'}} (H_i, 0) \times (0, e_{j'}) \in H \otimes H \end{aligned}$$

in a neighborhood of  $W$ , and this for  $Q_\nu^{(m,r)}$ -a.e.  $W \in \Omega$  and all  $s \in \mathcal{R}$ , implies the first part of (iii).

Proof. By condition (iii) of this proposition it follows now that for every  $\varepsilon > 0$  there is  $\rho_0 \equiv \rho_0(W, \varepsilon)$  such that

$$\begin{aligned} &\left| \frac{X_s(W^\rho) - X_s(W)}{\rho} - \left\langle D_F X_s(W), \frac{j^{-1}(W^\rho - W)}{\rho} \right\rangle_H \right. \\ &\quad \left. - \frac{1}{2} \left\langle \left\langle D_F^2 X_s(W), \frac{j^{-1}(W^\rho - W)}{\rho} \right\rangle_H, j^{-1}(W^\rho - W) \right\rangle_H \right| \\ &\leq \varepsilon \frac{\|j^{-1}(W^\rho - W)\|_H^2}{|\rho|}, \quad |\rho| \leq \rho_0. \end{aligned} \quad (2.31)$$

For the notation, cf. also Subsection 1.1, in particular the comment following condition (5). Using condition (iv) of this proposition it follows from (2.31) that

$$\frac{d^\pm}{d\rho} \Big|_{\rho=0} X_s(W^\rho) = \frac{d^\pm}{d\rho} \Big|_{\rho=0} \langle D_F X_s(W), j^{-1} W^\rho \rangle_H.$$

Taking into consideration conditions (i) and (ii) of the present proposition we get for  $Q_\nu^{(m,r)}$ -a.e.  $W \in \Omega$

$$\begin{aligned} \frac{d^\pm}{d\rho} \Big|_{\rho=0} X_s(W^\rho) &= \frac{d^\pm}{d\rho} \Big|_{\rho=0} \langle D_F X_s(W), j^{-1} f(\rho, W) \rangle_H \\ &= \left\langle D_F X_s(W), j^{-1} \dot{f}(0, W) \right\rangle_{H \rightarrow F}. \end{aligned} \quad (2.32)$$

In addition, we note that  $D_F X_s = \nabla_H X_s$  by (iii) of the present proposition. Accordingly, the right-hand side of (2.32) is by (i) of the present proposition an element of  $L^1(\Omega, Q_\nu^{(m,r)}; F)$  whenever  $\nabla_H X_s \in L^q(\Omega, Q_\nu^{(m,r)}; H)$ .  $\square$

**Stochastic integral in a distributional sense.** Let us assume  $\dot{f}(0, \cdot)(s) \in D_{p,1}$ ,  $s \in \mathcal{R}$ . We have

$$\left\langle H_i, d\dot{f}(0, \cdot) \right\rangle_{L^2} \in D_{p,1}, \quad i \in I(m, r),$$

by the special shape of the  $H_i$ . Consequently

$$D \left\langle H_i, d\dot{f}(0, W) \right\rangle_{L^2} \in H, \quad i \in I(m, r),$$

for  $Q_\nu^{(m,r)}$ -a.e.  $W \in \Omega$ . With this we may redefine (2.12)  $Q_\nu^{(m,r)}$ -a.e. by

$$\begin{aligned} \hat{\delta}(j^{-1}\dot{f}(0, \cdot))(W) &:= \sum_{i \in I(m,r)} \left\langle H_i, d\dot{f}(0, W) \right\rangle_{L^2} \cdot \langle H_i, dW \rangle_{L^2} \\ &\quad - \sum_{i \in I(m,r)} \left\langle (H_i, 0), D \left\langle H_i, d\dot{f}(0, W) \right\rangle_{L^2} \right\rangle_H \\ &\quad - \left\langle \dot{f}(0, W)(0), \frac{\nabla m(W_0)}{m(W_0)} \right\rangle_F - \left\langle \mathbf{e}, \nabla_{d, W_0} \dot{f}(0, W)(0) \right\rangle_F \end{aligned} \quad (2.33)$$

where we note that we just have shown the well-definiteness of the right-hand side.

The left-hand side of (2.33) is only a symbol reminding of the identity (2.12) demonstrated under the more restrictive setting of Subsection 2.2. But this will change under the specification of Section 3. In fact, having proved the subsequent Lemmas 3.2 and 3.3, the right-hand side of (2.33) will become the object that plays the role of a stochastic integral.

### 3 The Density Formula

Besides the proof of Theorem 1.11, this section contains a number of technical preparations. There are two major technical ideas in order to prove Theorem 1.11. The first one is to re-map  $X$  to  $W$ , to carry out the analysis on Wiener process related objects and then to map the result obtained there back to a result in terms of  $X$ . The second idea is to work on orthogonal projections of paths of  $W$  rather than on the trajectories of  $W$  itself. In Subsection 3.1 we will specify the flow  $W^\rho := W_{+\rho} + (A_\rho - Y_\rho)\mathbb{1}$ ,  $\rho \in \mathbb{R}$ , and provide technical calculations relative to the finite dimensional projections of this flow. In Subsection 3.2 we will be interested in the stochastic integral relative to (2.33). Then, in Subsection 3.3, we will collect a number of technical details of the approximation of the flow  $W^\rho = W_{+\rho} + (A_\rho - Y_\rho)\mathbb{1}$ ,  $\rho \in \mathbb{R}$ . The actual proof of Theorem 1.11 will be presented in Subsection 3.4.

Subsections 3.1 and 3.4 are entirely preparatory for the proof of Theorem 1.11 whereas Subsections 3.2 and 3.3 provide some more insight on the related stochastic calculus.

#### 3.1 Specification of $f$

One goal of this subsection is to specify  $f$  in (2.5), and with this also  $h$ , to those functionals we are going to use in the proof of Theorem 1.11. In particular, we will consider the flow

$$X(W^\rho) = u(W^\rho) = u(W_{+\rho} + (A_\rho - Y_\rho)\mathbb{1}) = X(W_{+\rho} + (A_\rho - Y_\rho)\mathbb{1}), \quad \rho \in \mathbb{R}. \quad (3.1)$$

Correspondingly, we also will suppose

$$f(\rho, W) := W_{+\rho} + (A_\rho - Y_\rho)\mathbb{1} - W \quad (3.2)$$

which includes that  $W^\rho = W + f(\rho, W) = W_{+\rho} + (A_\rho - Y_\rho)\mathbb{1}$ ,  $\rho \in \mathbb{R}$ , is a flow, cf. Remark (3) of Section 1. In the Lemmas 3.1 and 3.2 we will assume (3.1) and (3.2). Let us show that  $f$  is compatible with the analysis of Subsection 2.3 and, in particular, satisfies the conditions (i) and (ii) of Proposition 2.7. Let us also recall that the space  $D_{p',1} \equiv D_{p',1}(Q_\nu^{(m,r)})$  is meaningfully defined for  $p' \geq Q/(Q-1)$ , cf. Definition 1.1 and Proposition 6.3.

**Lemma 3.1** Assume conditions (1'), (2), and (3) of Section 1 which include in particular (3.1) as well as (3.2). Furthermore, assume  $A = A^1$  and  $Y = Y^1$ . Let  $q$  be the number introduced in condition (1') of Section 1 and suppose  $Y_\rho \in D_{q,1} \equiv D_{q,1}(Q_\nu^{(m,r)})$ ,  $\rho \in \mathbb{R}$ .

(a) For  $s \in \mathcal{R}$ ,  $i \in I(m, r)$ , and  $\rho \in \mathbb{R}$ , we have  $W_s, \langle H_i, d\dot{W} \rangle_{L^2} \in \bigcap_{p' \in [Q/(Q-1), \infty)} D_{p',1}$  and  $W_{s+\rho} + A_\rho - Y_\rho \in D_{q,1}$  where  $q$  is given by condition (1') of Section 1. Furthermore, it holds that

$$DW_s = \sum_{i \in I(m,r)} \int_0^s H_i(v) dv \cdot (H_i, 0) + (0, \mathbf{e}) \text{ for } Q_\nu^{(m,r)}\text{-a.e. } W \in \Omega \text{ and that}$$

$$D\langle H_i, d\dot{W} \rangle_{L^2} = \sum_{j \in I(m,r)} \langle H_i, dH_j \rangle_{L^2} \cdot (H_j, 0) \text{ for } Q_\nu^{(m,r)}\text{-a.e. } W \in \Omega.$$

(b)  $(W^\rho)_{\rho \in \mathbb{R}}$  satisfies (2.6) and (2.7).

(c) Let  $1/p + 1/q = 1$ . If  $(\dot{A}_\rho - \dot{Y}_\rho) \in L^p(\Omega, Q_\nu^{(m,r)}; F)$  then we have (i) and (ii) of Proposition 2.7, the latter demonstrating the compatibility of the weak mixed derivative in (2.7) with the integral of Definition 2.6.

(d) We have (iv) of Proposition 2.7.

Proof. (a) This is straight forward by the following. By Lemma 2.2 (a) we have

$$DW_s = \sum_{i \in I(m,r)} \frac{\partial W_s}{\partial \kappa_i} \cdot (H_i, 0) + (0, \nabla_{W_0} W_s) = \sum_{i \in I(m,r)} \int_0^s H_i(v) dv \cdot (H_i, 0) + (0, \mathbf{e}).$$

Recalling (2.2) we get under the measure  $Q_\nu^{(m,r)}$

$$D\langle H_i, d\dot{W} \rangle_{L^2} = \sum_{j \in I(m,r)} \frac{\partial \langle H_i, d\dot{W} \rangle_{L^2}}{\partial \kappa_j} \cdot (H_j, 0) = \sum_{j \in I(m,r)} \langle H_i, dH_j \rangle_{L^2} \cdot (H_j, 0).$$

Thus  $W_s, \langle H_i, d\dot{W} \rangle_{L^2} \in \bigcap_{p' \in [Q/(Q-1), \infty)} D_{p',1}$  is now clear, in particular by the bounded support of  $m$ , cf. Subsection 1.2.

$W_{s+\rho} + A_\rho - Y_\rho \in D_{q,1}$  is a consequence of  $A_\rho = X_\rho - W_\rho$  as well as  $X_\rho \in D_{q,1}$  by condition (1') of Section 1 and  $Y_\rho \in D_{q,1}$  by hypothesis.

(b) Let us verify (2.6). By definition,  $W^\sigma = W^\rho$  for some  $\sigma \neq \rho$  would imply  $W_{\cdot+\sigma} - W_{\cdot+\rho} = (A_\rho(W) - Y_\rho(W) - A_\sigma(W) + Y_\sigma(W)) \mathbb{I}$ . Assuming this, the right-hand side was  $Q_\nu^{(m,r)}$ -a.e. constant in time but the left-hand side was not.

Let us focus on (2.7). We have  $f(\rho, W)(s) = W_{s+\rho} + (A_\rho - Y_\rho) \mathbb{I}(s) - W_s$  which means in the setting of (2.7)

$$\begin{aligned} f^0(\rho, W)(s) &= W_{s+\rho} - W_s - W_\rho + W_0, \\ B_\rho &= W_\rho - W_0 + A_\rho - Y_\rho. \end{aligned}$$

Keeping (2.8) in mind we shall verify (2.7) with

$$\begin{aligned} \dot{f}(\rho, W)(s) &:= \sum_{i \in I(m,r)} \langle H_i, dW \rangle_{L^2} \cdot H_i(s + \rho) + \dot{A}_\rho - \dot{Y}_\rho \\ &= \sum_{i \in I(m,r)} \langle H_i, dW \rangle_{L^2} \cdot \left( H_i(s + \rho) - \frac{1}{2} (H_i(\rho-) + H_i(\rho)) \right) + \dot{W}_\rho + \dot{A}_\rho - \dot{Y}_\rho, \end{aligned} \tag{3.3}$$

$s \in \mathcal{R}$ ,  $\rho \in \mathbb{R}$ . We have

$$\begin{aligned}
& \frac{1}{\lambda} (\langle g, df^0(\rho + \lambda, W) \rangle_{L^2} - \langle g, df^0(\rho, W) \rangle_{L^2}) + \frac{1}{\lambda} (\langle x, B_{\rho+\lambda} \rangle_F - \langle x, B_\rho \rangle_F) \\
&= \sum_{i \in I(m, r)} \langle H_i, dW \rangle_{L^2} \cdot \frac{1}{\lambda} \left( \left\langle g, H_i(\cdot + \rho + \lambda) - \frac{1}{2} (H_i(\lambda + \rho -) + H_i(\lambda + \rho)) \cdot \mathbf{1} \right\rangle_{L^2} \right. \\
&\quad \left. - \left\langle g, H_i(\cdot + \rho) - \frac{1}{2} (H_i(\rho -) + H_i(\rho)) \cdot \mathbf{1} \right\rangle_{L^2} \right) \\
&\quad + \frac{1}{\lambda} \left( \langle x, (W_{\rho+\lambda} + A_{\rho+\lambda} - Y_{\rho+\lambda}) - (W_\rho + A_\rho - Y_\rho) \rangle_F \right)
\end{aligned}$$

and therefore

$$\begin{aligned}
& \frac{d^\pm}{d\rho} \langle g, df^0(\rho, W) \rangle_{L^2} + \frac{d^\pm}{d\rho} \langle x, B_\rho \rangle_F \\
&= \sum_{i \in I(m, r)} \langle H_i, dW \rangle_{L^2} \cdot \langle g, dH_i(\cdot + \rho) \rangle_{L^2} + \left\langle x, \dot{W}_\rho + \dot{A}_\rho - \dot{Y}_\rho \right\rangle_F \\
&= \left\langle g, df(\rho, W) \right\rangle_{L^2} + \left\langle x, \dot{W}_\rho + \dot{A}_\rho - \dot{Y}_\rho \right\rangle_F, \tag{3.4}
\end{aligned}$$

$k = (g, x) \in H$ ,  $g : \mathcal{R} \rightarrow F$  cadlag of finite jump variation,  $x \in F$ ,  $\rho \in \mathbb{R}$   $Q_\nu^{(m, r)}$ -a.e., as a direct consequence of (3.3). In other words, we have (2.7).

(c) For condition (i) of Proposition 2.7,  $\dot{f}(\rho, \cdot) \in I^p$ , we take into consideration (3.3). In particular,  $(\dot{f}(\rho, \cdot))_a = (\dot{W}_\rho + \dot{A}_\rho - \dot{Y}_\rho) \mathbf{1}$ . Recall also  $(\dot{A}_\rho - \dot{Y}_\rho) \in L^p(\Omega, Q_\nu^{(m, r)}; F)$  by hypothesis. We get

$$j^{-1} \left( \dot{f}(\rho, \cdot) \right)_a = (0, \dot{W}_\rho + \dot{A}_\rho - \dot{Y}_\rho) \in L^p(\Omega, Q_\nu^{(m, r)}; H).$$

Furthermore,  $\sum_{i \in I(m, r)} \langle H_i, dW \rangle_{L^2} \cdot (H_i(\cdot + \rho) - \frac{1}{2} (H_i(\rho -) + H_i(\rho)))$  is a cadlag pure jump process with  $Q_\nu^{(m, r)}$ -a.e. finitely many jumps whose jump times are non-random and whose magnitudes in the sense of Definition 1.4 (iii) belong to  $L^p(\Omega, Q_\nu^{(m, r)})$ . Thus we have verified that  $\dot{f}(\rho, \cdot) \in I^p$ .

For condition (ii) of Proposition 2.7, we recall now (3.4) and the calculation before (3.4) which now say

$$\left\langle k, j^{-1} \dot{f}(\rho, W) \right\rangle_H = \frac{d^\pm}{d\rho} \langle g, df^0(\rho, W) \rangle_{L^2} + \left\langle x, \dot{B}_\rho \right\rangle_F,$$

the left-hand side in the sense of Definition 2.6. For compatibility with this definition, we restrict ourselves here to all test elements  $k = (g, x) \in H$  with  $g = g_c + g_j$  where  $g_c \in C(\mathcal{R}; F)$  and  $g_j : \mathcal{R} \rightarrow F$  being a cadlag pure jump function of finite jump variation and  $x \in F$ .

(d) This follows directly from

$$j^{-1} f(\rho, \cdot)(s) = \left( \sum_{i \in I(m, r)} \langle H_i, dW \rangle_{L^2} \cdot (H_i(s + \rho) - H_i(s)), W_\rho + A_\rho - Y_\rho - W_0 \right),$$

hypotheses  $A = A^1$  and  $Y = Y^1$  together with conditions (2) (i) and (3) (iii) of Section 1, and  $A_0 = Y_0$ .  $\square$

In order to prepare the crucial equalities in the proof of Theorem 1.11, namely (3.110) and (3.111) below, let us prove the subsequent lemma.

For this, let  $\varphi$  be a cylindrical function on  $C(\mathbb{R}; F)$  of the form  $\varphi(W) = f_0(W_0) \cdot f_1(W_{t_1} - W_0, \dots, W_{t_k} - W_0)$ ,  $W \in \Omega$ , where  $f_0 \in C_b^1(F)$ ,  $f_1 \in C_b^1(F^k)$ , and  $k \in \mathbb{N}$  such that  $t_l \in \mathcal{R} \cap \{z \cdot t / 2^m : z \in \mathbb{Z} \setminus \{0\}\}$ ,  $l \in \{1, \dots, k\}$ . We will also use the abbreviation  $(D_F - D)(W_\sigma + A_\sigma - Y_\sigma) := D_F(W_\sigma + A_\sigma - Y_\sigma) - D(W_\sigma + A_\sigma - Y_\sigma)$  and the following integrals. Let  $v \in [0, t]$ .

$$\begin{aligned} L_1 &:= \int \int_{\sigma=0}^v \left\langle D\varphi \circ (W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{I}), j^{-1} \left( \dot{W} + \left( \dot{A}_0 - \dot{Y}_0 \right) \mathbb{I} \right) \right\rangle_H d\sigma Q_\nu^{(m,r)}(dW) \\ L_2 &:= \int \int_{\sigma=0}^v \left( f_1(W_{t_1+\sigma} - W_\sigma, \dots, W_{t_k+\sigma} - W_\sigma) \left\langle \nabla f_0(W_\sigma + A_\sigma - Y_\sigma), \right. \right. \\ &\quad \left. \left. (D_F - D)(W_\sigma + A_\sigma - Y_\sigma) \right\rangle_{F \rightarrow F}, j^{-1} \left( \dot{W} + \left( \dot{A}_0 - \dot{Y}_0 \right) \mathbb{I} \right) \right\rangle_H \Big) d\sigma Q_\nu^{(m,r)}(dW) \end{aligned}$$

$$\begin{aligned} R_1 &:= \sum_{i \in I(r)} \int_{\sigma=0}^v \int \varphi([W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{I}] \circ \pi_{m,r}) \cdot \langle H_i, d([W] \circ \pi_{m,r}) \rangle_{L^2} \times \\ &\quad \times \langle H_i, dW \rangle_{L^2} Q_\nu(dW) d\sigma \end{aligned}$$

$$\begin{aligned} R_2 &:= - \sum_{i \in I(r)} \int_{\sigma=0}^v \int \varphi([W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{I}] \circ \pi_{m,r}) \times \\ &\quad \times \left\langle (H_i, 0), D_F \langle H_i, d([W] \circ \pi_{m,r}) \rangle_{L^2} \right\rangle_H Q_\nu(dW) d\sigma \end{aligned}$$

$$J := \int_{\sigma=0}^v \int \left\langle D\varphi \circ (W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{I}), \left( 0, \dot{W}_0 + \dot{A}_0 - \dot{Y}_0 \right) \right\rangle_H Q_\nu^{(m,r)}(dW) d\sigma.$$

Well-definiteness of these integrals will be demonstrated in parts (d)-(f) of the following lemma.

**Lemma 3.2** *Assume conditions (1'), (2), and (3) of Section 1 which in particular imply (3.1) as well as (3.2). Let  $q$  be the number introduced in condition (1') of Section 1. Furthermore, suppose  $Y_\rho \in D_{q,1}(Q_\nu^{(m,r)})$ ,  $\rho \in \mathbb{R}$ . Also assume (iii) of Proposition 2.7 for  $X$  as well as for  $X$  replaced with  $Y$ .*

(a) *We have*

$$\varphi \circ (W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{I}) \in D_{q,1}, \quad \sigma \in [0, t],$$

*with respect to the measure  $Q_\nu^{(m,r)}$ .*

(b) *Assume  $A = A^1$  and  $Y = Y^1$   $Q_\nu^{(m,r)}$ -a.e. For  $v \in [0, t]$  we have  $Q_\nu^{(m,r)}$ -a.e.*

$$\begin{aligned} &\varphi(W_{\cdot+v} + (A_v - Y_v)\mathbb{I}) - \varphi(W) \\ &= \int_{\sigma=0}^v \left\langle (D\varphi)(\pi_{m,r}(W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{I})), j^{-1}(\pi_{m,r}(W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{I})) \right\rangle_H d\sigma. \end{aligned}$$

(c) Assume  $A = A^1$  and  $Y = Y^1$   $Q_\nu^{(m,r)}$ -a.e. and let  $\sigma \in [0, t]$ . We have  $Q_\nu^{(m,r)}$ -a.e.

$$\begin{aligned} & \left\langle (D\varphi) \left( \pi_{m,r} (W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}) \right), j^{-1} \left( \pi_{m,r} (W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}) \right) \cdot \right\rangle_H \\ &= \left\langle D\varphi \circ (W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}), j^{-1} \left( \dot{W} + (\dot{A}_0 - \dot{Y}_0)\mathbb{1} \right) \right\rangle_H \\ &+ f_1(W_{t_1+\sigma} - W_\sigma, \dots, W_{t_k+\sigma} - W_\sigma) \left\langle \nabla f_0(W_\sigma + A_\sigma - Y_\sigma), \right. \\ & \left. (D_F - D)(W_\sigma + A_\sigma - Y_\sigma) \right\rangle_{F \rightarrow F}, j^{-1} \left( \dot{W} + (\dot{A}_0 - \dot{Y}_0)\mathbb{1} \right) \right\rangle_H. \end{aligned}$$

For the well-definiteness of these integral terms recall Definition 2.6.

(d) Let the assumptions of part (c) be in force. Suppose  $(\dot{A}_\rho - \dot{Y}_\rho) \in L^p(\Omega, Q_\nu^{(m,r)}; F)$ ,  $\rho \in \mathbb{R}$ , where  $1/p + 1/q = 1$ .

For  $\sigma \in [0, t]$ , let  $\nabla_H X_\sigma \in L^q(\Omega, Q_\nu^{(m,r)}; H)$  and  $\nabla_H Y_\sigma \in L^q(\Omega, Q_\nu^{(m,r)}; H)$ . Suppose that the corresponding norms belong, with respect to  $\sigma \in [0, t]$ , to  $L^1([0, t])$ . The above defined integrals  $L_1, L_2, R_1, R_2, J$  are well-defined and we have

$$L_1 + L_2 = R_1 + R_2 + J.$$

(e) For all  $i \in I(m, r)$ , we have

$$\varphi(W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}) \langle H_i, d\dot{W} \rangle_{L^2} \in D_{q',1}, \quad \sigma \in [0, t]$$

if  $Q/(Q-1) < q' < q < \infty$ . Furthermore,  $(H_i, 0) \in \text{Dom}_{p'}(\delta)$  for all  $i \in I(m, r)$  and all  $p < p' < Q$  and  $\delta(H_i, 0) = \langle H_i, dW \rangle_{L^2} \in \bigcap_{p' \in [1, \infty)} L^{p'}(\Omega, Q_\nu^{(m,r)})$  as well as

$$R_1 = \sum_{i \in I(m,r)} \int_{\sigma=0}^v \int \varphi(W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}) \cdot \left\langle H_i, d\dot{W} \right\rangle_{L^2} \langle H_i, dW \rangle_{L^2} Q_\nu^{(m,r)}(dW) d\sigma$$

and

$$R_2 = - \sum_{i \in I(m,r)} \int_{\sigma=0}^v \int \varphi(W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}) \cdot \left\langle D \left\langle H_i, d\dot{W} \right\rangle_{L^2}, (H_i, 0) \right\rangle_H Q_\nu^{(m,r)}(dW) d\sigma.$$

(f) Assume  $A = A^1$  and  $Y = Y^1$   $Q_\nu^{(m,r)}$ -a.e. and let  $Q$  be the number that is used in the definition of the density  $m$ . Let  $1/p + 1/q = 1$  with  $p \geq q$  and assume  $(\dot{A}_\rho - \dot{Y}_\rho) \in L^p(\Omega, Q_\nu^{(m,r)}; F)$ ,  $\rho \in \mathbb{R}$ . Suppose the existence of the derivative

$$\nabla_{a, W_0} (\dot{A}_0 - \dot{Y}_0) \quad \text{in} \quad L^q(\Omega, Q_\nu^{(m,r)}; F).$$

We have  $(0, \dot{W}_0 + \dot{A}_0 - \dot{Y}_0) \in \text{Dom}_w(\delta)$  with  $w = \min(q, (1/p + 1/Q)^{-1})$  in the sense of Subsection 1.2 and

$$\delta(0, \dot{W}_0 + \dot{A}_0 - \dot{Y}_0) = - \left\langle \dot{W}_0 + \dot{A}_0 - \dot{Y}_0, \frac{\nabla m(W_0)}{m(W_0)} \right\rangle_F - \left\langle \mathbf{e}, \nabla_{a, W_0} (\dot{A}_0 - \dot{Y}_0) \right\rangle_F$$

$\in L^w(\Omega, Q_\nu^{(m,r)})$ . Equivalently, we have

$$\begin{aligned} & \int \left\langle \dot{W}_0 + \dot{A}_0 - \dot{Y}_0, (D\psi)_2(W) \right\rangle_F Q_\nu^{(m,r)}(dW) \\ &= - \int \left( \left\langle \dot{W}_0 + \dot{A}_0 - \dot{Y}_0, \frac{\nabla m(W_0)}{m(W_0)} \right\rangle_F + \left\langle \mathbf{e}, \nabla_{d,W_0} (\dot{A}_0 - \dot{Y}_0) \right\rangle_F \right) \cdot \psi(W) Q_\nu^{(m,r)}(dW) \end{aligned} \quad (3.5)$$

for all  $\psi \in D_{v,1}$  where  $1/v + 1/w = 1$ . In particular,

$$J = \int \int_{\sigma=0}^v \varphi(W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}) \delta(0, \dot{W}_0 + \dot{A}_0 - \dot{Y}_0) d\sigma Q_\nu^{(m,r)}(dW).$$

Proof. (a) Actually, everything to prove can be taken from the differential calculus of real functions in finite dimensional domains. However, the following more extensive presentation will also be a preparation of the proof of part (c). Keeping Definition 1.1, (6.5), (6.6), Lemma 2.2 (a), and the special choice of the  $t_l$  in mind, we obtain

$$\begin{aligned} \langle (D\varphi)(W), j^{-1}\rho \rangle_H &= f_0(W_0) \cdot \sum_{l=1}^k \langle \nabla_l f_1(W_{t_1} - W_0, \dots, W_{t_k} - W_0), \rho_{t_l} - \rho_0 \rangle_F \\ &\quad + f_1(W_{t_1} - W_0, \dots, W_{t_k} - W_0) \cdot \langle \nabla f_0(W_0), \rho_0 \rangle_F \end{aligned} \quad (3.6)$$

with  $\rho \equiv (\rho - \rho_0 \cdot \mathbb{1}, \rho_0) \in \{j(f, x) : (f, x) \in H\}$  and  $\nabla_l$  denoting the gradient with respect to the  $l$ -th entry. Accordingly,

$$\begin{aligned} D\varphi \circ (W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}) &= f_0(W_\sigma + A_\sigma - Y_\sigma) \times \\ &\quad \times \sum_{l=1}^k \langle \nabla_l f_1(W_{t_1+\sigma} - W_\sigma, \dots, W_{t_k+\sigma} - W_\sigma), D(W_{t_l+\sigma} - W_\sigma) \rangle_{F \rightarrow F} \\ &\quad + f_1(W_{t_1+\sigma} - W_\sigma, \dots, W_{t_k+\sigma} - W_\sigma) \times \\ &\quad \times \langle \nabla f_0(W_\sigma + A_\sigma - Y_\sigma), D(W_\sigma + A_\sigma - Y_\sigma) \rangle_{F \rightarrow F}. \end{aligned} \quad (3.7)$$

Now, by Lemma 3.1 (a), we have  $\varphi \circ (W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}) \in D_{q,1}$ .

(b) We use the fact that, because of its special form, we have  $\varphi \circ \pi_{m,r} = \varphi$ . Thus  $Q_\nu^{(m,r)}$ -a.e. it holds that

$$\begin{aligned} & \varphi(W_{\cdot+v} + (A_v - Y_v)\mathbb{1}) - \varphi(W) \\ &= \varphi \circ \pi_{m,r}(W_{\cdot+v} + (A_v - Y_v)\mathbb{1}) - \varphi(W) \\ &= \int_{\sigma=0}^v \frac{d^\pm}{d\sigma} \varphi \circ \pi_{m,r}(W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}) d\sigma \\ &= \int_{\sigma=0}^v \left\langle (D\varphi)(\pi_{m,r}(W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1})), j^{-1}(\pi_{m,r}(W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1})) \right\rangle_H d\sigma. \end{aligned}$$

Here we have also applied Lemma 2.2 (a), the special form of  $H_i$ ,  $i \in I(m, r)$ , and the differential calculus similar as in the proof of Proposition 2.7.

(c) By Lemma 3.1 (a) we have  $Q_\nu^{(m,r)}$ -a.e.

$$\begin{aligned} D(W_{t_l+\sigma} - W_\sigma) &= \sum_{i \in I(m,r)} \frac{\partial(W_{t_l+\sigma} - W_\sigma)}{\partial \kappa_i} \cdot (H_i, 0) \\ &= \sum_{i \in I(m,r)} \int_\sigma^{t_l+\sigma} H_i(s) ds \cdot (H_i, 0), \quad l \in \{1, \dots, k\}, \end{aligned}$$

which is by the definition of  $I(m, r)$  also the orthogonal projection in  $H$  of  $(\mathbf{e}\mathbb{I}_{[\sigma, t_l+\sigma]}, 0)$  to the subspace spanned by the functions  $(\mathbb{I}_{[z \cdot t/2^m, (z+1) \cdot t/2^m]} \cdot e_j, 0)$ ,  $z \in \mathbb{Z}$ ,  $j \in \{1, \dots, n \cdot d\}$ . With

$$\sigma^-(m) \equiv \sigma^-(m, \sigma) := \max\{z \cdot t/2^m : z \in \mathbb{Z}, z \leq \sigma\}$$

we obtain  $Q_\nu^{(m,r)}$ -a.e. for  $l \in \{1, \dots, k\}$

$$\begin{aligned} D(W_{t_l+\sigma} - W_\sigma) &= (\mathbf{e}\mathbb{I}_{[\sigma^-(m), t_l+\sigma^-(m)]}, 0) \\ &\quad + 2^m \frac{\sigma - \sigma^-(m)}{t} \cdot \left( (\mathbf{e}\mathbb{I}_{[t_l+\sigma^-(m), t_l+\sigma^-(m)+t/2^m]}, 0) - (\mathbf{e}\mathbb{I}_{[\sigma^-(m), \sigma^-(m)+t/2^m]}, 0) \right). \end{aligned}$$

Recalling now the definition of the *weak mixed derivative* and taking Definition 2.6 into consideration it turns out that  $Q_\nu^{(m,r)}$ -a.e.

$$\begin{aligned} \left\langle D(W_{t_l+\sigma} - W_\sigma), j^{-1} \left( \dot{W} + \left( \dot{A}_0 - \dot{Y}_0 \right) \mathbb{I} \right) \right\rangle_{H \rightarrow F} &= \left\langle D(W_{t_l+\sigma} - W_\sigma), j^{-1} \dot{W} \right\rangle_{H \rightarrow F} \\ &= \frac{1}{2} \left( W'_{t_l+\sigma^-(m)} - W'_{t_l+\sigma^-(m)-t/2^m} \right) - \frac{1}{2} \left( W'_{\sigma^-(m)} - W'_{\sigma^-(m)-t/2^m} \right) \\ &\quad + 2^{m-1} \frac{\sigma - \sigma^-(m)}{t} \cdot \left( W'_{t_l+\sigma^-(m)+t/2^m} - 2W'_{t_l+\sigma^-(m)} + W'_{t_l+\sigma^-(m)-t/2^m} \right) \\ &\quad - 2^{m-1} \frac{\sigma - \sigma^-(m)}{t} \cdot \left( W'_{\sigma^-(m)+t/2^m} - 2W'_{\sigma^-(m)} + W'_{\sigma^-(m)-t/2^m} \right) \\ &= \frac{1}{2} \left( (p_{m,r} W'_{\cdot+\sigma})_{t_l} - (p_{m,r} W'_{\cdot+\sigma})_{t_l-t/2^m} \right) - \frac{1}{2} \left( (p_{m,r} W'_{\cdot+\sigma})_0 - (p_{m,r} W'_{\cdot+\sigma})_{-t/2^m} \right) \\ &=: \frac{1}{2} \Delta \left( p_{m,r} \dot{W}_{\cdot+\sigma} \right)_{t_l} - \frac{1}{2} \Delta \left( p_{m,r} \dot{W}_{\cdot+\sigma} \right)_0, \quad l \in \{1, \dots, k\}, \end{aligned}$$

here, as in Subsection 1.2 introduced,  $W'$  denoting the right derivative. It follows from Proposition 2.7, Lemma 3.1 (c), (d), and condition (3) of Section 1 that

$$\begin{aligned} \dot{W}_\sigma + \dot{A}_\sigma &\equiv \frac{d}{d\rho} \Big|_{\rho=0}^\pm X_{\sigma+\rho}(W) = \frac{d}{d\rho} \Big|_{\rho=0}^\pm X_\sigma(W^\rho) \\ &= \left\langle D_F(W_\sigma + A_\sigma), j^{-1} \left( \dot{W} + \left( \dot{A}_0 - \dot{Y}_0 \right) \mathbb{I} \right) \right\rangle_{H \rightarrow F} \quad Q_\nu^{(m,r)}\text{-a.e.} \end{aligned}$$

Furthermore, by condition (3) of Section 1 it holds that  $Y_{\sigma+\rho}(W) = A_0(W^{\sigma+\rho}) = A_0((W^\rho)^\sigma) = Y_\sigma(W^\rho)$ ,  $\rho \in \mathbb{R}$ ,  $\sigma \in [0, t]$ . Similar to the above we get therefore

$$\begin{aligned} \dot{Y}_\sigma &\equiv \frac{d}{d\rho} \Big|_{\rho=0}^\pm Y_{\sigma+\rho}(W) = \frac{d}{d\rho} \Big|_{\rho=0}^\pm Y_\sigma(W^\rho) \\ &= \left\langle D_F(Y_\sigma), j^{-1} \left( \dot{W} + \left( \dot{A}_0 - \dot{Y}_0 \right) \mathbb{I} \right) \right\rangle_{H \rightarrow F} \quad Q_\nu^{(m,r)}\text{-a.e.} \end{aligned}$$

After these preparations we obtain

$$\begin{aligned}
& \sum_{l=1}^k f_0(W_\sigma + A_\sigma - Y_\sigma) \cdot \left\langle \langle \nabla_l f_1(W_{t_1+\sigma} - W_\sigma, \dots, W_{t_k+\sigma} - W_\sigma), \right. \\
& \quad \left. D(W_{t_l+\sigma} - W_\sigma) \rangle_{F \rightarrow F}, j^{-1} \left( \dot{W} + \left( \dot{A}_0 - \dot{Y}_0 \right) \mathbb{I} \right) \right\rangle_H \\
& + f_1(W_{t_1+\sigma} - W_\sigma, \dots, W_{t_k+\sigma} - W_\sigma) \times \\
& \times \left\langle \langle \nabla f_0(W_\sigma + A_\sigma - Y_\sigma), D_F(W_\sigma + A_\sigma - Y_\sigma) \rangle_{F \rightarrow F}, j^{-1} \left( \dot{W} + \left( \dot{A}_0 - \dot{Y}_0 \right) \mathbb{I} \right) \right\rangle_H \\
& = \sum_{l=1}^k \left\langle f_0(W_\sigma + A_\sigma - Y_\sigma) \cdot \nabla_l f_1(W_{t_1+\sigma} - W_\sigma, \dots, W_{t_k+\sigma} - W_\sigma), \right. \\
& \quad \left. \frac{1}{2} \Delta \left( p_{m,r} \dot{W}_{\cdot+\sigma} \right)_{t_l} - \frac{1}{2} \Delta \left( p_{m,r} \dot{W}_{\cdot+\sigma} \right)_0 \right\rangle_F \\
& + \left\langle \nabla f_0(W_\sigma + A_\sigma - Y_\sigma) \cdot f_1(W_{t_1+\sigma} - W_\sigma, \dots, W_{t_k+\sigma} - W_\sigma), \dot{W}_\sigma + \dot{A}_\sigma - \dot{Y}_\sigma \right\rangle_F \\
& = \left\langle (D\varphi) \left( \pi_{m,r} \left( W_{\cdot+\sigma} + (A_\sigma - Y_\sigma) \mathbb{I} \right) \right), j^{-1} \left( \pi_{m,r} \left( W_{\cdot+\sigma} + (A_\sigma - Y_\sigma) \mathbb{I} \right) \right) \cdot \right\rangle_H
\end{aligned}$$

$Q_\nu^{(m,r)}$ -a.e. The claim follows now from (3.6), (3.7), and Definition 2.6.

(d) Using the notation  $(D_{F,1})_s \xi(W) = \sum_{i \in I(r)} \frac{\partial \xi(W)}{\partial \kappa_i} \cdot H_i(s)$ ,  $s \in \mathcal{R}$ , the following calculation is straight forward.

$$\begin{aligned}
L_1 + L_2 &= \int \int_{\sigma=0}^v \left\langle D\varphi \circ (W_{\cdot+\sigma} + (A_\sigma - Y_\sigma) \mathbb{I}), j^{-1} \left( \dot{W} + \left( \dot{A}_0 - \dot{Y}_0 \right) \mathbb{I} \right) \right\rangle_H d\sigma Q_\nu^{(m,r)}(dW) \\
& + \int \int_{\sigma=0}^v \left( f_1(W_{t_1+\sigma} - W_\sigma, \dots, W_{t_k+\sigma} - W_\sigma) \left\langle \langle \nabla f_0(W_\sigma + A_\sigma - Y_\sigma), \right. \right. \\
& \quad \left. \left. (D_F - D)(W_\sigma + A_\sigma - Y_\sigma) \rangle_{F \rightarrow F}, j^{-1} \left( \dot{W} + \left( \dot{A}_0 - \dot{Y}_0 \right) \mathbb{I} \right) \right\rangle_H \right) d\sigma Q_\nu^{(m,r)}(dW) \\
& = \int \int_{\sigma=0}^v \sum_{l=1}^k f_0(W_\sigma + A_\sigma - Y_\sigma) \cdot \left\langle \langle \nabla_l f_1(W_{t_1+\sigma} - W_\sigma, \dots, W_{t_k+\sigma} - W_\sigma), \right. \\
& \quad \left. D_1(W_{t_l+\sigma} - W_\sigma) \rangle_{F \rightarrow F}, d\dot{W} \right\rangle_{L^2} d\sigma Q_\nu^{(m,r)}(dW) \\
& + \int \int_{\sigma=0}^v f_1(W_{t_1+\sigma} - W_\sigma, \dots, W_{t_k+\sigma} - W_\sigma) \times \\
& \quad \times \left\langle \langle \nabla f_0(W_\sigma + A_\sigma - Y_\sigma), D_{F,1}(W_\sigma + A_\sigma - Y_\sigma) \rangle_{F \rightarrow F}, d\dot{W} \right\rangle_{L^2} d\sigma Q_\nu^{(m,r)}(dW) \\
& + \int \int_{\sigma=0}^v \left\langle D\varphi \circ (W_{\cdot+\sigma} + (A_\sigma - Y_\sigma) \mathbb{I}), \left( 0, \dot{W}_0 + \dot{A}_0 - \dot{Y}_0 \right) \right\rangle_H d\sigma Q_\nu^{(m,r)}(dW).
\end{aligned}$$

All integrals are well-defined by part (a), the hypotheses to the present part (d), and Proposition 2.7.

We will now regard the first two integrals on the right-hand side as integrals with respect to the measure  $Q_\nu$  rather than  $Q_\nu^{(m,r)}$ . As compensation the integrands will be considered

under the projection  $\pi_{m,r}$ . We obtain

$$\begin{aligned}
L_1 + L_2 &= \int \int_{\sigma=0}^v \sum_{l=1}^k f_0([W_\sigma + A_\sigma - Y_\sigma] \circ \pi_{m,r}) \cdot \left\langle \nabla_l f_1([W_{t_l+\sigma} - W_\sigma] \circ \pi_{m,r}, \dots, \right. \\
&\quad \left. [W_{t_k+\sigma} - W_\sigma] \circ \pi_{m,r}), D_1([W_{t_l+\sigma} - W_\sigma] \circ \pi_{m,r}) \right\rangle_{F \rightarrow F}, \\
&\quad d([W] \circ \pi_{m,r}) \cdot \Big\rangle_{L^2} d\sigma Q_\nu(dW) \\
&+ \int \int_{\sigma=0}^v f_1([W_{t_1+\sigma} - W_\sigma] \circ \pi_{m,r}, \dots, [W_{t_k+\sigma} - W_\sigma] \circ \pi_{m,r}) \times \\
&\quad \times \left\langle \nabla f_0([W_\sigma + A_\sigma - Y_\sigma] \circ \pi_{m,r}), D_{F,1}([W_\sigma + A_\sigma - Y_\sigma] \circ \pi_{m,r}) \right\rangle_{F \rightarrow F}, \\
&\quad d([W] \circ \pi_{m,r}) \cdot \Big\rangle_{L^2} d\sigma Q_\nu(dW) + J \\
&= \int \int_{\sigma=0}^v \sum_{i \in I(r)} \left\langle D_{F,1} \varphi \circ ([W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}] \circ \pi_{m,r}), H_i \right\rangle_{L^2} \times \\
&\quad \times \left\langle H_i, d([W] \circ \pi_{m,r}) \cdot \right\rangle_{L^2} d\sigma Q_\nu(dW) + J \\
&=: r_1 + J
\end{aligned}$$

We note that  $r_1$  is independent of the measure  $\nu$ . Thus,  $r_1$  is an integral with respect to the classical Wiener measure  $Q(dV) := \int_{W_0 \in D^n} Q_\nu(d(V + W_0\mathbb{1}))$  where  $W = V + W_0\mathbb{1} \in \Omega$  and  $V \in \Omega_0 := \{U \in \Omega : U_0 = 0\}$ . Therefore,  $D_{F,1}$  is the Malliavin derivative with respect to  $Q$  on  $(\Omega_0, \mathcal{B}(\Omega_0))$  and the corresponding Sobolev norm in  $D_{q,1}$ . We obtain

$$\begin{aligned}
L_1 + L_2 &= \sum_{i \in I(r)} \int_{\sigma=0}^v \int \varphi([W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}] \circ \pi_{m,r}) \cdot \left\langle H_i, d([W] \circ \pi_{m,r}) \cdot \right\rangle_{L^2} \times \\
&\quad \times \left\langle H_i, dW \right\rangle_{L^2} Q_\nu(dW) d\sigma \\
&- \sum_{i \in I(r)} \int_{\sigma=0}^v \int \varphi([W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}] \circ \pi_{m,r}) \times \\
&\quad \times \left\langle (H_i, 0), D_F \left\langle H_i, d([W] \circ \pi_{m,r}) \cdot \right\rangle_{L^2} \right\rangle_H Q_\nu(dW) d\sigma + J.
\end{aligned}$$

(e) For  $\langle H_i, d\dot{W} \rangle_{L^2} \in \bigcap_{r \in [Q/(Q-1), \infty)} D_{r,1}$ , recall Lemma 3.1 (a),  $i \in I(m, r)$ . Together with part (a) this implies  $\varphi \circ (W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}) \langle H_i, d\dot{W} \rangle_{L^2} \in D_{q',1}$ ,  $\sigma \in [0, t]$ , for  $Q/(Q-1) < q' < q$  and  $i \in I(m, r)$ . We have

$$\begin{aligned}
R_1 &= \sum_{i \in I(m,r)} \int_{\sigma=0}^v \int \varphi(W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}) \cdot \left\langle H_i, d\dot{W} \right\rangle_{L^2} \langle H_i, dW \rangle_{L^2} Q_\nu^{(m,r)}(dW) d\sigma \\
&+ \sum_{i \in I(r) \setminus I(m,r)} \int_{\sigma=0}^v \int \varphi([W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}] \circ \pi_{m,r}) \cdot \left\langle H_i, d([W] \circ \pi_{m,r}) \cdot \right\rangle_{L^2} \times
\end{aligned}$$

$$\begin{aligned}
& \times \langle H_i, dW \rangle_{L^2} Q_\nu(dW) d\sigma \\
& = \sum_{i \in I(m,r)} \int_{\sigma=0}^v \int \varphi(W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}) \cdot \left\langle H_i, d\dot{W} \right\rangle_{L^2} \langle H_i, dW \rangle_{L^2} Q_\nu^{(m,r)}(dW) d\sigma
\end{aligned}$$

since  $\varphi([W]_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}) \circ \pi_{m,r} \cdot \langle H_i, d([W] \circ \pi_{m,r}) \rangle_{L^2}$  is for  $i \in I(r) \setminus I(m,r)$  independent of  $\langle H_i, dW \rangle_{L^2}$  and  $\int \langle H_i, dW \rangle_{L^2} Q_\nu(dW) = 0$ . Furthermore, for  $i \in I(r) \setminus I(m,r)$  we have  $\langle H_i, d([W] \circ \pi_{m,r}) \rangle_{L^2} = 0$ . Thus

$$\begin{aligned}
R_2 & = - \sum_{i \in I(m,r)} \int_{\sigma=0}^v \int \varphi([W]_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}) \circ \pi_{m,r} \times \\
& \quad \times \left\langle (H_i, 0), D \langle H_i, d([W] \circ \pi_{m,r}) \rangle_{L^2} \right\rangle_H Q_\nu(dW) d\sigma \\
& = - \sum_{i \in I(m,r)} \int_{\sigma=0}^v \int \varphi(W_{\cdot+\sigma} + (A_\sigma - Y_\sigma)\mathbb{1}) \times \\
& \quad \times \left\langle (H_i, 0), D \left\langle H_i, d\dot{W} \right\rangle_{L^2} \right\rangle_H Q_\nu^{(m,r)}(dW) d\sigma.
\end{aligned}$$

(f) For  $\varphi$  cylindrical in the sense of this lemma, we have

$$\begin{aligned}
& \int \left\langle (0, \dot{W}_0 + \dot{A}_0 - \dot{Y}_0), D\varphi(W) \right\rangle_H Q_\nu^{(m,r)}(dW) \\
& = \int \int \left\langle \dot{W}_0 + \dot{A}_0 - \dot{Y}_0, \nabla_{W_0} \varphi(W) \right\rangle_F Q_{W_0}^{(m,r)}(dW) m(W_0) dW_0 \\
& = \int \left\langle \int (\dot{W}_0 + \dot{A}_0 - \dot{Y}_0) \cdot f_1(W_{t_1} - W_0, \dots, W_{t_k} - W_0) Q_{W_0}^{(m,r)}(dW), \right. \\
& \quad \left. \nabla_{W_0} f_0(W_0) \right\rangle_F m(W_0) dW_0.
\end{aligned}$$

We observe that  $f_1(W_{t_1} - W_0, \dots, W_{t_k} - W_0) Q_{W_0}^{(m,r)}(dW)$  is independent of  $W_0$  and obtain

$$\begin{aligned}
& \int \left\langle (0, \dot{W}_0 + \dot{A}_0 - \dot{Y}_0), D\varphi(W) \right\rangle_H Q_\nu^{(m,r)}(dW) \\
& = - \int \left( \left\langle \dot{W}_0 + \dot{A}_0 - \dot{Y}_0, \frac{\nabla m(W_0)}{m(W_0)} \right\rangle_F + \left\langle \mathbf{e}, \nabla_{d,W_0} (\dot{A}_0 - \dot{Y}_0) \right\rangle_F \right) \cdot \varphi(W) Q_\nu^{(m,r)}(dW)
\end{aligned}$$

where we have taken into consideration  $\nabla_{W_0} \dot{W}_0 = 0$ . For the well-definiteness we refer to the definition of the density  $m$ , condition (1') of Subsection 1.2, as well as to  $(\dot{A}_\rho - \dot{Y}_\rho) \in L^p(\Omega, Q_\nu^{(m,r)}; F)$  where  $p \geq q$  by hypothesis. Moreover, we recall the existence of  $\nabla_{d,W_0}(\dot{A}_0 - \dot{Y}_0)$  in  $L^q(\Omega, Q_\nu^{(m,r)}; F)$  and  $q \geq w$  by hypothesis. The last chain of equation yields  $(0, \dot{W}_0 + \dot{A}_0 - \dot{Y}_0) \in \text{Dom}_w(\delta)$  and the stochastic integral  $\delta(0, \dot{W}_0 + \dot{A}_0 - \dot{Y}_0)$  in the sense of Subsection 1.2.  $\square$

### 3.2 Related stochastic integral

The following lemma explains the stochastic integral in (2.33) and provides detailed insight in the crucial step of the proof of Theorem 1.11, relation (3.111).

**Lemma 3.3** *Let  $Q$  be the number appearing in the definition of  $m$  in Subsection 1.2, let  $q$  be the number of condition (1') in Section 1, and let  $1/p + 1/q = 1$  with  $p \geq q$ .*

*Assume conditions (1'), (2), and in particular,*

$$W^\rho = f(\rho, W) + W = W_{\cdot+\rho} + (A_\rho - Y_\rho)\mathbb{I}, \quad \rho \in \mathbb{R}. \quad (3.8)$$

*Assume furthermore*

$$(i) \quad (\dot{A}_0 - \dot{Y}_0) \in L^p(\Omega, Q_\nu^{(m,r)}; F) \text{ and } \nabla_{d,W_0} (\dot{A}_0 - \dot{Y}_0) \in L^q(\Omega, Q_\nu^{(m,r)}; F),$$

$$(ii) \quad A = A^1, Y = Y^1.$$

*We have  $(0, \dot{W}_0 + \dot{A}_0 - \dot{Y}_0) \in \text{Dom}_w(\delta)$  with  $w = \min(q, (1/p + 1/Q)^{-1})$  in the sense of Subsection 1.2 and for the term (2.33)*

$$\begin{aligned} \hat{\delta} \left( j^{-1} \dot{f}(0, \cdot) \right) &= \delta \left( 0, \dot{W}_0 + \dot{A}_0 - \dot{Y}_0 \right) \\ &= - \left\langle \dot{W}_0 + \dot{A}_0 - \dot{Y}_0, \frac{\nabla m(W_0)}{m(W_0)} \right\rangle_F - \left\langle \mathbf{e}, \nabla_{d,W_0} (\dot{A}_0 - \dot{Y}_0) \right\rangle_F \end{aligned}$$

*for  $Q_\nu^{(m,r)}$ -a.e.  $W \in \Omega$ .*

**Remarks.** (1) For the term (2.33) we had assumed  $\dot{f}(0, \cdot)(s) \in D_{p,1}$ ,  $s \in \mathcal{R}$ , following the context of Section 2. Now that we have specified  $f$  and  $\dot{f}$  (cf. (3.2) and (3.3)) it is natural to suppose  $(\dot{A}_0 - \dot{Y}_0) \in L^p(\Omega, Q_\nu^{(m,r)}; F)$ .

(2) By condition (1') in Subsection 1.2 and  $p \geq q$  we have  $w > 1$ . By (i) of this lemma and the definition of  $m$  in Subsection 1.2 the first item of the right-hand side of the above representation of  $\hat{\delta} \left( j^{-1} \dot{f}(0, \cdot) \right)$  belongs to  $L^w(\Omega, Q_\nu^{(m,r)})$ . By hypothesis we have  $q \geq w$ . Hence the second item of  $\hat{\delta} \left( j^{-1} \dot{f}(0, \cdot) \right)$  belongs also to  $L^w(\Omega, Q_\nu^{(m,r)})$ .

Proof. We show that the right-hand side of (2.33) forms a stochastic integral in the sense of Subsection 1.2. First, we observe that the right-hand side of (2.33),

$$\begin{aligned} I(W) &:= \sum_{i \in I(m,r)} \left\langle H_i, d\dot{f}(0, W) \right\rangle_{L^2} \cdot \langle H_i, dW \rangle_{L^2} - \sum_{i \in I(m,r)} \left\langle (H_i, 0), D \left\langle H_i, d\dot{f}(0, W) \right\rangle_{L^2} \right\rangle_H \\ &\quad - \left\langle \dot{f}(0, W)(0), \frac{\nabla m(W_0)}{m(W_0)} \right\rangle_F - \left\langle \mathbf{e}, \nabla_{d,W_0} \dot{f}(0, W)(0) \right\rangle_F, \end{aligned}$$

simplifies significantly because of

$$\sum_{i \in I(m,r)} \left\langle H_i, d\dot{f}(0, W) \right\rangle_{L^2} \cdot \langle H_i, dW \rangle_{L^2} = \sum_{i \in I(m,r)} \langle H_i, dW \rangle_{L^2} \cdot \left\langle H_i, d\dot{W} \right\rangle_{L^2} = 0, \quad (3.9)$$

and

$$\begin{aligned} - \sum_{i \in I(m,r)} \left\langle (H_i, 0), D \left\langle H_i, d\dot{f}(0, W) \right\rangle_{L^2} \right\rangle_H &= - \sum_{i \in I(m,r)} \left\langle (H_i, 0), D \left\langle H_i, d\dot{W} \right\rangle_{L^2} \right\rangle_H \\ &= - \sum_{i \in I(m,r)} \langle H_i, dH_i \rangle_{L^2} \\ &= 0 \quad \text{for } Q_\nu^{(m,r)}\text{-a.e. } W \in \Omega. \end{aligned} \quad (3.10)$$

In fact, relation (3.9) follows from (3.8) and (2.3) and relation (3.10) is a consequence of (3.8), Lemma 3.1 (a), and the definition of the integral  $\langle H_i, dH_i \rangle_{L^2}$  in Subsection 2.1. We obtain

$$\begin{aligned} I(W) &= - \left\langle \dot{f}(0, W)(0), \frac{\nabla m(W_0)}{m(W_0)} \right\rangle_F - \left\langle \mathbf{e}, \nabla_{d, W_0} \dot{f}(0, W)(0) \right\rangle_F \\ &= - \left\langle \dot{W}_0 + \dot{A}_0 - \dot{Y}_0, \frac{\nabla m(W_0)}{m(W_0)} \right\rangle_F - \left\langle \mathbf{e}, \nabla_{d, W_0} (\dot{A}_0 - \dot{Y}_0) \right\rangle_F = 0 \end{aligned} \quad (3.11)$$

for  $Q_\nu^{(m,r)}$ -a.e.  $W \in \Omega$ , where we have taken into consideration  $\nabla_{d, W_0} \dot{W}_0 = 0$ . The lemma follows now from (3.11) and Lemma 3.2 (f), especially relation (3.5).  $\square$

### 3.3 Technical details of the approximation

In this section we carry out all calculations for the approximation of trajectories with jumps,  $A$ ,  $X$ , and  $Y$ , by continuous trajectories  $A_n$ ,  $X_n$ , and  $Y_n$ . Furthermore, we present the technicalities for the extension of the analysis on  $[-rt, (r-1)t]$  of the trajectories to the analysis of the trajectories on  $\mathbb{R}$ . For the readers convenience we collect all claims in the subsequent Lemma 3.4. The proofs of all individual statements follow then.

**Lemma 3.4** *Assume conditions (1)-(5) of Section 1. For all  $W \in \Omega$ , there exist sequences of processes  $A_n(W)$  and  $Y_n(W)$  such that with  $X_n(W) := W + A_n(W)$ ,  $n \in \mathbb{N}$ , the following holds.*

(i)  $A_n = A_n^1$ ,  $Y_n = Y_n^1$ ,  $n \in \mathbb{N}$ , and moreover  $A_n = A_n^1$  and  $Y_n = Y_n^1$  are continuously differentiable.

(ii) Condition (iii) of Proposition 2.7 holds for  $X_n$  as well as  $Y_n$ . Moreover, for  $s \in \mathbb{R}$ ,

$$\nabla_H X_{n,s} \in \bigcap_{1 \leq p < \infty} L^p(\Omega, Q_\nu^{(m,r)}; H) \quad \text{and} \quad \nabla_H Y_{n,s} \in \bigcap_{1 \leq p < \infty} L^p(\Omega, Q_\nu^{(m,r)}; H).$$

In addition, we have  $A_{n,s}, Y_{n,s} \in D_{q,1}(Q_\nu^{(m,r)})$  for all  $q$  with  $1/q + 1/Q < 1$  where  $Q$  is the number introduced in the definition of the density  $m$  in Subsection 1.2.

(iii) We have

$$\dot{A}_{n,s} - \dot{Y}_{n,s} \in \bigcap_{1 \leq p < \infty} L^p(\Omega, Q_\nu^{(m,r)}; F) \cap L^p(\Omega, Q_\nu^{(m)}; F), \quad s \in \mathbb{R}.$$

Furthermore,  $\dot{A}_n(W) - \dot{Y}_n(W)$  is bounded on finite subintervals of  $\mathbb{R}$  for all  $W \in \bigcup_m \{\pi_m V : V \in \Omega\}$ .

(iv) For all  $p \in [1, \infty)$ , the derivative  $\nabla_{W_0} (\dot{A}_{n,0} - \dot{Y}_{n,0})$  exists in  $L^p(\Omega, Q_\nu^{(m,r)}; F)$ . In addition, for all  $p \in [1, \infty)$ ,  $\nabla_{W_0} (\dot{A}_{n,s} - \dot{Y}_{n,s})$ , seen as a limit in  $L^p(\Omega, Q_\nu^{(m)}; F)$ , exists uniformly for all  $s$  belonging to any finite subinterval of  $\mathbb{R}$ . In particular,  $\nabla_{W_0} (\dot{A}_n(W) - \dot{Y}_n(W))$  is bounded on finite subintervals  $\mathbb{R}$  for all  $W \in \bigcup_m \{\pi_m V : V \in \Omega\}$ . Furthermore,

$$\int_S \left| \left\langle \mathbf{e}, \nabla_{W_0} (\dot{A}_{n,s} - \dot{Y}_{n,s}) \right\rangle_F \right| ds \in \bigcap_{1 \leq p < \infty} L^p(\Omega, Q_\nu^{(m)})$$

for every finite subinterval  $S \subset \mathbb{R}$ .

(v) On  $\bigcup_m \{\pi_m W : W \in \Omega\}$ ,  $\nabla_{W_0} (A_n - Y_n)$  is continuously differentiable on  $\mathbb{R}$  and we have  $\nabla_{W_0} (\dot{A}_{n,s} - \dot{Y}_{n,s}) = (\nabla_{W_0} (A_{n,s} - Y_{n,s}))'$ .

(vi) With  $W^{n,u} := W_{\cdot+u} + (A_{n,u}(W) - Y_{n,u}(W)) \mathbb{1}$ ,  $u \in \mathbb{R}$ , the process  $X_n = W + A_n(W)$  is temporally homogeneous on  $W \in \Omega$ ; i.e., we have

$$W^{n,0} = W, \quad A_{n,0}((W^{n,u})^v) = A_{n,0}(W^{n,u+v}),$$

and

$$X_{n,\cdot+v}(W) = X_n(W^{n,v}), \quad v \in \mathbb{R}.$$

Furthermore,  $Y_{n,v}(W) = A_{n,0}(W^{n,v})$ ,

$$Y_{n,\cdot+v}(W) = Y_n(W^{n,v}), \quad \text{and} \quad \dot{A}_{n,\cdot+v}(W) = \dot{A}_n(W^{n,v}), \quad v \in \mathbb{R}.$$

In addition,  $A_{n,\cdot}$  and  $Y_{n,\cdot}$  are constant on intervals where  $W_0^{n,\cdot} \notin \{x \in F : |x - z| < \frac{1}{n^3}, z \in D^n\}$ .

We have the following convergences.

(a)  $A_{n,s}(W) - Y_{n,s}(W) \xrightarrow{n \rightarrow \infty} A_s(W) - Y_s(W)$  for all  $W \in \bigcup_m \{\pi_m V : V \in \Omega\}$  with  $W_0 \notin G(W - W_0 \mathbb{1})$ , cf. Subsection 1.2, whenever neither 0 nor  $s$  is a jump time for  $X$ .

(b) Let  $W \in \{\pi_m V : V \in \Omega\}$  with  $W_0 \notin G(W - W_0 \mathbb{1})$ . Then

$$A_{n,s}(\pi_{m,r} W) - Y_{n,s}(\pi_{m,r} W) \xrightarrow{r \rightarrow \infty} A_{n,s}(W) - Y_{n,s}(W),$$

$$\dot{A}_{n,s}(\pi_{m,r} W) - \dot{Y}_{n,s}(\pi_{m,r} W) \xrightarrow{r \rightarrow \infty} \dot{A}_{n,s}(W) - \dot{Y}_{n,s}(W),$$

as well as

$$\nabla_{W_0} \dot{A}_{n,s}(\pi_{m,r} W) - \nabla_{W_0} \dot{Y}_{n,s}(\pi_{m,r} W) \xrightarrow{r \rightarrow \infty} \nabla_{W_0} \dot{A}_{n,s}(W) - \nabla_{W_0} \dot{Y}_{n,s}(W), \quad s \in \mathbb{R}.$$

Furthermore,

$$\sup_{r \in \mathbb{N}} \left| \dot{A}_{n,s}(\pi_{m,r} W) - \dot{Y}_{n,s}(\pi_{m,r} W) \right| \in \bigcap_{1 \leq p < \infty} L^p(\Omega, Q_\nu^{(m)}; F)$$

as well as

$$\sup_{r \in \mathbb{N}} \left| \nabla_{W_0} \dot{A}_{n,s}(\pi_{m,r} W) - \nabla_{W_0} \dot{Y}_{n,s}(\pi_{m,r} W) \right| \in L^1(\Omega, Q_\nu^{(m)}; F), \quad s \in \mathbb{R},$$

where the absolute values  $|\cdot|$  and the  $\sup_{r \in \mathbb{N}}$  are taken individually for all coordinates of  $F$ .

(c) For all  $-\infty < l < r < \infty$  it holds that

$$\begin{aligned} & \nabla_{W_0} ((A_{n,l} - Y_{n,l}) - (A_{n,r} - Y_{n,r})) \\ & \xrightarrow{n \rightarrow \infty} (\nabla_{W_0} A_l - \nabla_{W_0} Y_l) - (\nabla_{W_0} A_r - \nabla_{W_0} Y_r) \end{aligned}$$

uniformly bounded  $Q_\nu^{(m)}$ -a.e. on  $\{\pi_m W : W \in \Omega\}$ .

**Remarks. (3)** For part (c), we recall that according to conditions (2) and (3) of Subsection 1.2,  $A$  and  $Y$  jump only on  $Q_\nu^{(m)}$ -zero sets. Following condition (1) of Subsection 1.2 it turns out that  $Q_\nu^{(m)}$ -a.e. on  $\{\pi_m W : W \in \Omega\}$  the set of all of jump times of  $\nabla_{W_0} A$  and  $\nabla_{W_0} Y$  is a subset of  $\{\tau_k : k \in \mathbb{Z} \setminus \{0\}\}$ .

**(4)** In order to ease the notation in the proof below we will use the symbol  $|\cdot|$  to abbreviate  $\langle \cdot, \cdot \rangle_F^{1/2}$  in both cases,  $F = \mathbb{R}^{n \cdot d}$  as well as  $F = \mathbb{R}^{n \cdot d} \otimes \mathbb{R}^{n \cdot d}$ . Furthermore, the symbol  $|\cdot|$  will also be used for the coordinate wise absolute value. It will either be clear from the context or explicitly be mentioned which meaning the symbol has.

**(5)** For the sake of comprehensibility we first carry out the proofs of the parts (a) and (c) for the particular case that, for given  $W \in \Omega$  with  $W_0 = 0$  and  $x \in F$ , the jump times of  $X = u(W + x\mathbb{I})$  are independent of  $x \in F$ . In a separate paragraph at the end of this subsection, in we prove parts (a) and (c) of Lemma 3.4 if parallel trajectories in  $\Omega$  generate no longer identical jump times for  $X$ . There we consider jump times as introduced in Definition 1.7 of Subsection 1.2.

**Definition of  $A_n$  and  $Y_n$  and verification of (i) and (vi).** *Step 1* We define  $A_n$  and  $Y_n$  and construct the crucial system of ODEs.

Recall from Subsection 1.2 the definitions of  $g_n$ ,  $\gamma_n$ , and

$$A_s(\cdot, \gamma_n)(W) := \int_F \langle A_s(W + x\mathbb{I}), \gamma_n(x) \rangle_{F \rightarrow F} dx.$$

Introduce

$$A_s(g, \gamma_n)(W) := \int_{\mathbb{R}} \langle A_{s-v}(\cdot, \gamma_n)(W), g(v) \rangle_{F \rightarrow F} dv,$$

$s \in \mathbb{R}$ ,  $g \in \{f\mathbf{e} : f \in C(\mathbb{R})\}$ ,  $n \in \mathbb{N}$ ,  $W \in \Omega$ . Likewise define  $Y_s(g, \gamma_n)$  and  $Y_s(\cdot, \gamma_n)$ ,  $A'_s(g, \gamma_n)$  and  $A'_s(\cdot, \gamma_n), \dots$ . Furthermore, set

$$B_{n,s} := A_s(g_n, \gamma_n) \tag{3.12}$$

and

$$C_{n,s} := A_s(g_n, \gamma_n) - Y_s(g_n, \gamma_n), \quad s \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{3.13}$$

Let  $W \in \Omega$ . Let us consider the following system of first order ODEs

$$\begin{aligned} \dot{\varphi}(s) &= \dot{C}_{n,0}(\varphi(s)\mathbb{I} + W_{\cdot+s}) =: F(s, \varphi(s)), \quad s \in \mathbb{R}, \\ \varphi(0) &= 0. \end{aligned} \tag{3.14}$$

In particular, we have

$$\begin{aligned} F(s, x) &= \int_{y \in F} \int_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle \langle A_{-v}(W_{\cdot+s} + y\mathbb{I}) - Y_{-v}(W_{\cdot+s} + y\mathbb{I}), \right. \\ &\quad \left. g'_n(v) \rangle_{F \rightarrow F}, \gamma_n(y - x) \rangle_{F \rightarrow F} dv dy \\ &= \int_{y \in F} \int_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle \langle A'_{-v}(W_{\cdot+s} + y\mathbb{I}) - Y'_{-v}(W_{\cdot+s} + y\mathbb{I}), \right. \end{aligned}$$

$$\begin{aligned}
& g_n(v) \Big\rangle_{F \rightarrow F}, \gamma_n(y-x) \Big\rangle_{F \rightarrow F} dv dy \\
& + \int_F \sum_{k \in \mathbb{Z} \setminus \{0\}} \left\langle \left\langle \Delta A_{\tau_k}(W_{\cdot+s} + y \mathbb{I}) - \Delta Y_{\tau_k}(W_{\cdot+s} + y \mathbb{I}) \right\rangle \right. \\
& \quad \left. g_n(\tau_k \circ u(W_{\cdot+s} + y \mathbb{I})) \right\rangle_{F \rightarrow F}, \gamma_n(y-x) \Big\rangle_{F \rightarrow F} dy, \quad (3.15)
\end{aligned}$$

$s \in \mathbb{R}$ ,  $x \in F$ . It follows from Remark (6) of Section 1 and the definition of  $g_n$  as well as  $\gamma_n$  that, for fixed  $s_0 > 0$ ,  $F(s, x)$  is bounded on  $(s, x) \in [-s_0, s_0] \times F$ . Continuity of  $F(s, x)$  on  $(s, x) \in [-s_0, s_0] \times F$  is a consequence of condition (4) (i) of Section 1. Furthermore,  $F(s, \cdot)$  is continuously differentiable with uniformly bounded gradient for all  $s \in [-s_0, s_0]$ , cf. again Remark (6) of Section 1. Thus, the Picard-Lindelöf theorem can be applied to all intervals of the form  $[-s_0, s_0]$ . It follows that there exists a global unique solution  $\varphi \equiv \varphi(\cdot; n, W)$  to the equation (3.14).

*Step 2* We verify (i) and (vi). Equation (3.14) is equivalent to

$$\varphi(s) = \int_0^s \dot{C}_{n,0}(\varphi(v) \mathbb{I} + W_{\cdot+v}) dv \quad (3.16)$$

and also to

$$W_{\cdot+s} + \varphi(s) \mathbb{I} = W_{\cdot+s} + \int_0^s \dot{C}_{n,0}(\varphi(v) \mathbb{I} + W_{\cdot+v}) dv \mathbb{I}, \quad (3.17)$$

$s \in \mathbb{R}$ . We observe that by (3.17)

$$W^{n,s} := W_{\cdot+s} + \varphi(s) \mathbb{I}, \quad s \in \mathbb{R}, \quad (3.18)$$

is the unique solution to

$$W^{n,s} = W_{\cdot+s} + \int_0^s \dot{C}_{n,0}(W^{n,v}) dv \mathbb{I}, \quad s \in \mathbb{R}. \quad (3.19)$$

As a direct consequence,  $W^{n,s}$ ,  $s \in \mathbb{R}$ , is a flow which is continuous in the topology of uniform convergence on compact sets. Let us introduce

$$D_{n,s}(W) := \int_0^s \dot{C}_{n,0}(W^{n,v}) dv, \quad s \in \mathbb{R}. \quad (3.20)$$

From the flow property of  $W^{n,s}$ ,  $s \in \mathbb{R}$ , it follows that

$$D_{n,s+v}(W) = D_{n,s}(W) + D_{n,v}(W^{n,s}), \quad v \in \mathbb{R}. \quad (3.21)$$

We define

$$A_{n,s}(W) := D_{n,s}(W) + B_{n,0}(W^{n,s}) \quad \text{as well as} \quad Y_{n,s}(W) := B_{n,0}(W^{n,s}), \quad (3.22)$$

and

$$X_{n,s}(W) := W_s + A_{n,s}(W), \quad s \in \mathbb{R}.$$

The following becomes now obvious. By (3.19) and (3.22) we have

$$W_v^{n,s} = W_{s+v} + A_{n,s} - Y_{n,s}, \quad s, v \in \mathbb{R}. \quad (3.23)$$

Furthermore, we obtain from (3.21) and (3.22) and the flow property of  $W^{n,s}$ ,  $s \in \mathbb{R}$ ,

$$Y_{n,s}(W^{n,v}) = B_{n,0}(W^{n,s+v}) = Y_{n,v}(W^{n,s}) \quad (3.24)$$

as well as

$$\begin{aligned} A_{n,s+v}(W) - A_{n,s}(W) &= D_{n,s+v}(W) + B_{n,0}(W^{n,s+v}) - D_{n,s}(W) - B_{n,0}(W^{n,s}) \\ &= D_{n,v}(W^{n,s}) + B_{n,0}(W^{n,v}(W^{n,s})) - B_{n,0}(W^{n,s}) \\ &= A_{n,v}(W^{n,s}) - A_{n,0}(W^{n,s}), \quad s, v \in \mathbb{R}. \end{aligned} \quad (3.25)$$

As a consequence of (3.22) we obtain  $A_{n,0}(W^{n,s}) = Y_{n,s}(W)$ ,  $s \in \mathbb{R}$ . Together with (3.23) and (3.25) this leads to

$$\begin{aligned} X_{n,v}(W^{n,s}) &= W_v^{n,s} + A_{n,v}(W^{n,s}) \\ &= W_v^{n,s} + A_{n,0}(W^{n,s}) + A_{n,s+v}(W) - A_{n,s}(W) \\ &= W_{s+v} + A_{n,s}(W) - Y_{n,s}(W) + A_{n,0}(W^{n,s}) + A_{n,s+v}(W) - A_{n,s}(W) \\ &= W_{v+s} + A_{n,v+s}(W) \\ &= X_{n,v+s}(W), \quad s, v \in \mathbb{R}. \end{aligned} \quad (3.26)$$

Recall the definitions (3.18), (3.23). Taking in particular into consideration that  $\varphi(s) = A_{n,s} - Y_{n,s}$  is the unique solution to (3.14), it turns out that  $A_{n,s} - Y_{n,s}$  is continuously differentiable in  $s \in \mathbb{R}$ . Let us now demonstrate that  $Y_{n,s}$  and hence also  $A_{n,s}$  is continuously differentiable in  $s \in \mathbb{R}$ . Similar to (3.14) the system of first order ODEs

$$\begin{aligned} \dot{\xi}(s) &= \dot{B}_{n,0}((\eta(s) - \xi(s))\mathbb{I} + W_{\cdot+s}) \\ \dot{\eta}(s) &= \dot{C}_{n,0}((\eta(s) - \xi(s))\mathbb{I} + W_{\cdot+s}) + \dot{B}_{n,0}((\eta(s) - \xi(s))\mathbb{I} + W_{\cdot+s}), \quad s \in \mathbb{R}, \\ \xi(0) &= B_{n,0}(W) \\ \eta(0) &= C_{n,0}(W) + B_{n,0}(W). \end{aligned}$$

has for  $W \in \Omega$  a unique solution  $(\eta, \xi) \in C^1(\mathbb{R}; F^2)$ . We remark that the point wise derivative  $\dot{B}_{n,0}$  exists by the definition (3.12) and the convolutions in the terms above (3.12).

By (3.14) and (3.18) as well as (3.23), we have  $\eta(s) - \xi(s) = \varphi(s) = A_{n,s} - Y_{n,s}$ ,  $s \in \mathbb{R}$ . The first line of the above system and (3.22) as well as (3.24) imply

$$\dot{\xi}(s) = \dot{B}_{n,0}(W^{n,s}) = \dot{Y}_{n,0}(W^{n,s}) = \frac{d}{ds} Y_{n,s}(W), \quad s \in \mathbb{R}.$$

Thus  $Y_{n,s} \in C^1(\mathbb{R}; F)$  and therefore also  $A_{n,s} \in C^1(\mathbb{R}; F)$ . As a consequence, (3.25) results in

$$\dot{A}_{n,s+v}(W) = \dot{A}_{n,s}(W^{n,v}), \quad s, v \in \mathbb{R}.$$

Below, we will frequently use the calculation rules (3.24) and (3.25) together with its differential form displayed in the last relation.

Furthermore, we recall the above system of first order ODEs for  $\xi$  and  $\eta$  and the hypotheses  $A(W) \equiv 0$  and  $Y(W) \equiv 0$  whenever  $W_0 \notin D^n$ , cf. the definition of the process  $X$  in Subsection 1.2 and condition (3) (iii) of Section 1. It follows that  $A_{n,\cdot}$  and  $Y_{n,\cdot}$  are constant on intervals where  $W_0^{n,\cdot} \notin \{x \in F : |x - z| < \frac{1}{n^3}, z \in D^n\}$ .

We have defined  $A_n$  and  $Y_n$  by (3.12), (3.13), (3.20), and (3.22) and verified (i) and (vi) of the lemma.  $\square$

**Proof of part (a).** *Step 1* We intend to apply Gronwall's inequality for this. The inequality is stated in the end of Step 3. In this step we prepare the inequality by analyzing the structure of  $A_{n,s} - Y_{n,s} - (A_s - Y_s)$ .

In order to ease the notation we will frequently write  $(A - Y)(\cdot)$  for  $A(\cdot) - Y(\cdot)$ ,  $(A - Y)'$  for  $A' - Y'$ , and  $\Delta(A - Y)$  for  $\Delta A - \Delta Y$ . Fix  $W \in \{\pi_m V : V \in \Omega\}$  with  $W_0 \notin G(W - W_0 \mathbb{1})$ . It follows from (3.13), (3.19), (3.20), (3.22), and (3.23) that

$$\begin{aligned} A_{n,s}(W) - Y_{n,s}(W) &= \int_{r=0}^s \dot{C}_{n,0}(W^{n,r}) \, dr \\ &= \int_{r=0}^s \int_F \int \left\langle \left\langle (A - Y)'_{-v}(W^{n,r} + y \mathbb{1}), g_n(v) \right\rangle_{F \rightarrow F}, \gamma_n(y) \right\rangle_{F \rightarrow F} dv \, dy \, dr \\ &\quad + \int_{r=0}^s \int_F \sum_{k \in \mathbb{Z} \setminus \{0\}} \left\langle \left\langle \Delta(A - Y)_{\tau_k}(W^{n,r} + y \mathbb{1}), \right. \right. \\ &\quad \left. \left. g_n(\tau_k \circ u(W^{n,r} + y \mathbb{1})) \right\rangle_{F \rightarrow F}, \gamma_n(y) \right\rangle_{F \rightarrow F} dy \, dr, \end{aligned} \quad (3.27)$$

$s \in \mathbb{R}$ . We take into consideration Remark (5) of the present subsection which, in particular, implies  $\tau_k \circ u(W^{n,r} + y \mathbb{1}) = \tau_k \circ u(W^r)$  and set

$$\begin{aligned} \psi(r) &\equiv \psi(r; n, W) := A_{n,r}(W) - Y_{n,r}(W) - (A - Y)_r(W), \\ K(s, x) &\equiv K(s, x; n, W) \\ &:= \int_F \int \left\langle \left\langle (A - Y)'_{-v}(W^s + (x + y) \mathbb{1}) - (A - Y)'_{-v}(W^s), \right. \right. \\ &\quad \left. \left. g_n(v) \right\rangle_{F \rightarrow F}, \gamma_n(y) \right\rangle_{F \rightarrow F} dv \, dy \\ &\quad + \int_F \sum_{k \in \mathbb{Z} \setminus \{0\}} \left\langle \left\langle \Delta(A - Y)_{\tau_k}(W^s + (x + y) \mathbb{1}) - \Delta(A - Y)_{\tau_k}(W^s), \right. \right. \\ &\quad \left. \left. g_n(\tau_k \circ u(W^s)) \right\rangle_{F \rightarrow F}, \gamma_n(y) \right\rangle_{F \rightarrow F} dy \\ &\equiv K_1(s, x) + K_2(s, x), \end{aligned} \quad (3.28)$$

$$\rho(s) \equiv \rho(s; n, W)$$

$$\begin{aligned}
&:= \int_{r=0}^s \int \langle (A - Y)'_{-v}(W^r), g_n(v) \rangle_{F \rightarrow F} dv dr \\
&+ \int_{r=0}^s \sum_{k \in \mathbb{Z} \setminus \{0\}} \langle \Delta(A - Y)_{\tau_k}(W^r), g_n(\tau_k \circ u(W^r)) \rangle_{F \rightarrow F} dy dr \\
&-(A - Y)_s(W)
\end{aligned} \tag{3.29}$$

relation (3.27) results in

$$\psi(s) = \int_0^s K(v, \psi(v)) dv + \rho(s), \quad s \in \mathbb{R}. \tag{3.30}$$

*Step 2* We construct estimates on  $|\int_{r=0}^s K_i(r, \psi(r)) dr|$ ,  $i = 1, 2$ , and  $|\int_{r=0}^s K(r, \psi(r)) dr| + |\rho(s)|$ . Since  $\nabla_{W_0} A'_s$  and  $\nabla_{W_0} Y'_s$  exist in the sense of conditions (2) and (3) of Section 1,

$$\alpha'(S; W) := \sup \left| \nabla_x (A - Y)'_{-v} (W^s + x \mathbb{I}) \right| < \infty$$

where the supremum is taken over all  $(v, s, x) \in (-1, 1) \times S \times F$  and  $S \subset \mathbb{R}$  is any finite subinterval. It follows that

$$\left| \int_{r=0}^s K_1(r, \psi(r)) dr \right| \leq \alpha'(S; W) \int_{r=0}^s (|\psi(r)| + \frac{1}{n^3} \cdot \mathbf{e}) dr, \quad s \in S, \tag{3.31}$$

where the absolute value has been taken coordinate wise.

Since  $\Delta A$  and  $\Delta Y$  are piecewise continuously differentiable on  $F$ , cf. conditions (2) and (3) of Subsection 1.2,

$$\beta'(S; W) := \sup \left| \nabla_x \Delta(A - Y)_{\tau_k} (W^s + x \mathbb{I}) \right| < \infty$$

where the supremum is taken over all  $(k, s, x) \in \{l \in \mathbb{Z} \setminus \{0\} : \tau_l \in (-1, 1)\} \times S \times F$  and  $S \subset \mathbb{R}$  is any finite subinterval. Now, recall again that  $\Delta A$  and  $\Delta Y$  are piecewise continuously differentiable on  $F$ . In particular for any  $k \in \mathbb{Z} \setminus \{0\}$ , there exist finitely many mutually exclusive open sets  $F_1 \equiv F_1(k), F_2 \equiv F_2(k), \dots$  with piecewise  $C^1$ -boundary and  $\overline{F} = \bigcup_i \overline{F_i}$  such that  $\Delta A(W + \cdot \mathbb{I})|_{F_i}, \Delta Y(W + \cdot \mathbb{I})|_{F_i} \in C_b(F_i)$ .

The last relation together with  $\beta'(S; W) < \infty$  and condition (3) (ii) of Section 1 implies the existence of  $\gamma'(S; W) < \infty$  such that

$$\left| \Delta(A - Y)_{\tau_k} (W^s + x \mathbb{I}) - \Delta(A - Y)_{\tau_k} (W^s) \right| \leq \gamma'(S; W) \cdot |x|$$

for  $(k, s, x) \in \{l \in \mathbb{Z} \setminus \{0\} : \tau_l \in (-1, 1)\} \times S \times F$  and any finite subinterval  $S \subset \mathbb{R}$ . Also here, the absolute value has been taken coordinate wise. It follows that

$$\begin{aligned}
&\left| \int_{r=0}^s K_2(r, \psi(r)) dr \right| \\
&\leq \gamma'(S; W) \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{r=0}^s \left\langle |\psi(r)| + \frac{1}{n^3} \cdot \mathbf{e}, g_n(\tau_k \circ u(W^r)) \right\rangle_{F \rightarrow F} dr, \quad s \in S.
\end{aligned} \tag{3.32}$$

By  $X(W^r) = X_{\cdot+r}$ , cf. condition (3) of Section 1, we have

$$\tau_k \circ u(W^r) = \tau_{k'} \circ u(W) - r$$

for  $k \in \mathbb{Z} \setminus \{0\}$  and some  $k' \equiv k'(k; W, r) \in \mathbb{Z} \setminus \{0\}$  such that  $k - k'$  is a constant  $c \equiv c(W, r)$  if  $k$  and  $k'$  are both positive or both are negative. Moreover with the same constant  $c$ ,  $k - k' = c + 1$  if  $k > 0 > k'$  and  $k - k' = c - 1$  if  $k < 0 < k'$ , cf. the definition of the jumps in Subsection 1.2.

Relations (3.28)-(3.30) together with the estimates on  $\int_{r=0}^s K_i(r, \psi(r)) dr$ ,  $i = 1, 2$ , in (3.31) and (3.32), result in

$$\begin{aligned} |\psi(s)| &\leq \left| \int_0^s K(v, \psi(v)) dv \right| + |\rho(s)| \\ &\leq \alpha'(S; W) \int_{r=0}^s (|\psi(r)| + \frac{1}{n^3} \cdot \mathbf{e}) dr \\ &\quad + \gamma'(S; W) \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{r=0}^s \left\langle |\psi(r)| + \frac{1}{n^3} \cdot \mathbf{e}, g_n(\tau_k \circ u(W^r)) \right\rangle_{F \rightarrow F} dr + |\rho(s)| \\ &= \int_{r=0}^s \left\langle |\psi(r)|, \beta(r; n, W) \right\rangle_{F \rightarrow F} dr + R(s; n, W), \quad s \in S, \quad n \in \mathbb{N}, \end{aligned} \quad (3.33)$$

for any finite subinterval  $S \subset \mathbb{R}$  where all absolute values are coordinate wise,

$$R(s; n, W) := \frac{\alpha'(S; W) \cdot s}{n^3} \cdot \mathbf{e} + \frac{\gamma'(S; W)}{n^3} \sum_{k' \in \mathbb{Z} \setminus \{0\}} \int_{r=0}^s g_n(\tau_{k'} - r) dr + |\rho(s)|,$$

and

$$\beta(r; n, W) := \alpha'(S; W) \cdot \mathbf{e} + \gamma'(S; W) \sum_{k' \in \mathbb{Z} \setminus \{0\}} g_n(\tau_{k'} - r).$$

*Step 3* We apply Gronwall's inequality to carry out the proof of part (a) of Lemma 3.4. From condition (3) and Remark (3) of Section 1 we get

$$\begin{aligned} \rho(s; n, W) &= \int_{r=0}^s \int_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle (A - Y)'_{-v}(W^r), g_n(v) \right\rangle_{F \rightarrow F} dv dr \\ &\quad + \int_{r=0}^s \sum_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle \Delta(A - Y)_v(W^r), g_n(v) \right\rangle_{F \rightarrow F} dr - (A - Y)_s(W) \\ &= \int_{r=0}^s \int_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle (A - Y)'_{r-v}(W), g_n(v) \right\rangle_{F \rightarrow F} dv dr \\ &\quad + \int_{r=0}^s \sum_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle \Delta(A - Y)_{r-v}(W), g_n(v) \right\rangle_{F \rightarrow F} dr - (A - Y)_s(W) \\ &= \int_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle (A - Y)_{s-v}(W) - (A - Y)_{-v}(W), g_n(v) \right\rangle_{F \rightarrow F} dv \\ &\quad - (A - Y)_s(W) \end{aligned} \quad (3.34)$$

which implies

$$\rho(s; n, W) \xrightarrow{n \rightarrow \infty} 0$$

if neither 0 nor  $s$  is a jump time for  $X$ . With the last equality sign of (3.34) we get

$$\begin{aligned} & \langle \rho(s; n, W), \beta(s; n, W) \rangle_{F \rightarrow F} \\ &= \alpha'(S; W) \int_{v \in \mathbb{R}} \left\langle (A - Y)_{s-v}(W) - (A - Y)_s(W), g_n(v) \right\rangle_{F \rightarrow F} dv \\ &+ \gamma'(S; W) \sum_{k' \in \mathbb{Z} \setminus \{0\}} \int_{v \in \mathbb{R}} \left\langle (A - Y)_v(W) \left\langle g_n(s - v), g_n(\tau_{k'} - s) \right\rangle_{F \rightarrow F} \right\rangle_{F \rightarrow F} dv \\ &- \gamma'(S; W) \sum_{k' \in \mathbb{Z} \setminus \{0\}} \left\langle (A - Y)_s(W), g_n(\tau_{k'} - s) \right\rangle_{F \rightarrow F} \\ &- \left\langle \int_{v \in \mathbb{R}} \left\langle (A - Y)_{-v}(W), g_n(v) \right\rangle_{F \rightarrow F} dv, \beta(s; n, W) \right\rangle_{F \rightarrow F}. \end{aligned}$$

Keeping in mind  $A_0 = Y_0$  we obtain

$$\langle \rho(\cdot; n, W), \beta(\cdot; n, W) \rangle_{F \rightarrow F} \xrightarrow{n \rightarrow \infty} 0 \quad \text{in } L^1([0, s]; F)$$

if, without loss of generality,  $0 \leq s$  and neither 0 nor  $s$  is a jump time for  $X$ . It is a consequence of the definition of  $R(\cdot; n, W)$  and the fact that

$$\sum_{k' \in \mathbb{Z} \setminus \{0\}} \int_{r=0}^s g_n(\tau_{k'} - r) dr$$

is on every component uniformly bounded in  $n$  by the number of jumps of  $X = u(W)$  on  $(-1, s + 1]$  that

$$R(s; n, W) \xrightarrow{n \rightarrow \infty} 0 \tag{3.35}$$

$$\langle R(\cdot; n, W), \beta(\cdot; n, W) \rangle_{F \rightarrow F} \xrightarrow{n \rightarrow \infty} 0 \quad \text{in } L^1([0, s]; F)$$

if, without loss of generality,  $0 \leq s$  and neither 0 nor  $s$  are jump times for  $X$ . It follows now from (3.33) and Gronwall's inequality that

$$\begin{aligned} & \left| A_{n,s}(W) - Y_{n,s}(W) - (A_s(W) - Y_s(W)) \right| = |\psi(s)| \\ & \leq R(s; n, W) + \int_{v=0}^s \left\langle R(v; n, W), \beta(v; n, W) \right\rangle_{F \rightarrow F} \cdot \exp \left\{ \int_{r=v}^s |\beta(r; n, W)| dr \right\} dv \\ & \leq R(s; n, W) + \exp \left\{ \int_{r=0}^s |\beta(r; n, W)| dr \right\} \cdot \int_{v=0}^s \left\langle R(v; n, W), \beta(v; n, W) \right\rangle_{F \rightarrow F} dv, \end{aligned} \tag{3.36}$$

$n \in \mathbb{N}$ . Here the absolute values in the first line are coordinate wise, the absolute value on  $\beta$  is not, cf. Remark (4) of the present section. Part (a) follows now on the one hand from (3.35). On the other hand we use again the fact that  $\sum_{k' \in \mathbb{Z} \setminus \{0\}} \int_{r=0}^s g_n(\tau_{k'} - r) dr$  is on

every coordinate uniformly bounded in  $n$ . This implies that  $\int_{r=0}^s |\beta(r; n, W)| dr$  is uniformly bounded in  $n \in \mathbb{N}$  and we have accomplished the proof of (a).

As an addendum to this step but also as a preparation to other proofs of the present subsection we obtain the following. By (3.13) and (3.22) we have

$$\dot{A}_{n,s}(W) - \dot{Y}_{n,s}(W) = \int_{\mathbb{R}} \langle A_{-v}(\cdot, \gamma_n)(W^{n,s}) - Y_{-v}(\cdot, \gamma_n)(W^{n,s}), g'_n(v) \rangle_{F \rightarrow F} dv$$

and therefore

$$\left| \dot{A}_{n,s}(W) - \dot{Y}_{n,s}(W) \right| \leq \frac{2\|g'_n\|}{n} \sup_{v \in (-\frac{1}{n}, \frac{1}{n})} \left| A_{-v}(\cdot, \gamma_n)(W^{n,s}) - Y_{-v}(\cdot, \gamma_n)(W^{n,s}) \right| \quad (3.37)$$

$s \in \mathbb{R}$ ,  $n \in \mathbb{N}$ , where the absolute value  $|\cdot|$  has been taken individually for each coordinate of  $F$ .

We have  $\sup_{s,W} |Y_s(W)| < \infty$  by condition (3) of Section 1 and  $D^n$  is bounded by hypothesis. If  $W_0^{n,s} + x \notin D^n$  then, by hypothesis of Subsection 1.2,  $A(W^{n,s} + x\mathbb{I}) \equiv 0$ . On the other hand if  $W_0^{n,s} + x \in D^n$  then, again by hypothesis of Subsection 1.2, we may assume that  $X(W^{n,s} + x\mathbb{I}) \in D^n$  and take into consideration that  $A(W^{n,s} + x\mathbb{I}) = X(W^{n,s} + x\mathbb{I}) - (W_0^{n,s} + x) - (W_{+s} - W_s)$ .

In any case, by (3.37) there exists a positive constant  $c \equiv c(n)$  which may depend on  $n \in \mathbb{N}$  but is independent of  $s \in \mathbb{R}$  and  $W \in \{\pi_m V : V \in \Omega\}$  such that

$$\left| \dot{A}_{n,s}(W) - \dot{Y}_{n,s}(W) \right| \leq c(n) \mathbf{e} + \frac{2\|g'_n\|}{n} \sup_{v \in (-\frac{1}{n}, \frac{1}{n})} |W_{s-v} - W_s|. \quad (3.38)$$

□

**Proof of parts (c) and (v).** Let  $s \in \mathbb{R}$ ,  $n \in \mathbb{N}$ ,  $W \in \{\pi_m V : V \in \Omega\}$  with  $W_0 \notin G(W - W_0\mathbb{I})$ , and

$$\kappa(s) \equiv \kappa(s; n, W) := \nabla_{W_0} \rho(s; n, W), \quad (3.39)$$

for the well-definiteness of this expression for all  $s \in \mathbb{R}$  recall (3.34) and Remark (7) of Section 1. Let us also recall the existence of a local spatial gradient for  $A^1$  and  $Y^1$  and the piecewise continuous differentiability of the jump magnitudes, cf. conditions (2) as well as (3) and Remark (2) of Section 1. As a consequence of the definition of  $K$  in (3.28) the integral  $\int_0^s$  in (3.30) commutes with the derivative  $\nabla_{W_0}$ . Keeping these arguments in mind and noting that

$$\dot{C}_{n,0} = \int_{v \in (-\frac{2}{n}, 0)} \left\langle A_{-v}(\cdot, \gamma_n) - Y_{-v}(\cdot, \gamma_n), g'_n\left(v + \frac{1}{n}\right) \right\rangle_{F \rightarrow F} dv$$

we verify also that the integral  $\int_0^s$  in (3.16) commutes with the derivative  $\nabla_{W_0}$ . It follows now from (3.16) that  $\nabla_{W_0} \varphi \equiv \nabla_{W_0} \varphi(\cdot; n, W)$  satisfies, whenever it exists,

$$\begin{aligned} (\nabla_{W_0} \varphi(s))^\cdot &= \left( \nabla_{W_0} \dot{C}_{n,0} \right) (W^{n,s}) \cdot (\nabla_{W_0} \varphi(s) + \mathbf{e}), \quad s \in \mathbb{R}, \\ \nabla_{W_0} \varphi(0) &= 0 \end{aligned} \quad (3.40)$$

But existence of  $\nabla_{W_0}\varphi$  follows by the Picard-Lindelöf theorem. Together with (3.16) this yields, en passant, (v). More precisely, we note that  $\nabla_{W_0}\varphi$  is the unique solution to (3.40) by the representation

$$\nabla_{W_0}\dot{C}_{n,0} = \int_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle \nabla_{W_0} (A_{-v}(\cdot, \gamma_n)) - \nabla_{W_0} (Y_{-v}(\cdot, \gamma_n)), g'_n(v) \right\rangle_{F \rightarrow F} dv \quad (3.41)$$

and

$$\nabla_{W_0} (A_{-v}(\cdot, \gamma_n)(W^{n,s})) = - \int_F \left\langle A_{-v}(W^{n,s} + x\mathbb{I}) \mathbf{e}, \nabla_x \gamma_n(x) \right\rangle_{F \rightarrow F} dx, \quad (3.42)$$

the multiplication  $\langle \cdot, \cdot \rangle_{F \rightarrow F}$  taken here adequately coordinate wise; a similar representation holds for  $\nabla_{W_0} (Y_{-v}(\cdot, \gamma_n)(W^{n,s}))$ . In fact, the Picard-Lindelöf theorem yields uniqueness by the continuity of  $s \rightarrow W^{n,s}$  demonstrated in the proof of (i) of Lemma 3.4 which together with condition (4) (i) of Subsection 1.2 implies continuity of

$$s \rightarrow \left( \nabla_{W_0} \dot{C}_{n,0} \right) (W^{n,s}).$$

It follows from (3.30) that

$$\nabla_{W_0}\psi(s) = \int_0^s (\nabla_x K)(v, \psi(v)) \cdot \nabla_{W_0}\psi(v) dv + \kappa(s), \quad s \in \mathbb{R}, \quad (3.43)$$

where, for the well-definiteness of  $\nabla_{W_0}\psi(s)$  for all  $s \in \mathbb{R}$ , we first recall the well-definiteness of  $\nabla_{W_0}\varphi$ . Then, for  $\nabla_{W_0}\psi$  we keep Remark (7) of Section 1 in mind. According to (3.28) and Remark (5) of the present section we have also

$$\begin{aligned} K(s, x) = & \int_F \int \left\langle \left\langle (A - Y)'_{-v}(W^s + (x + y)\mathbb{I}) - (A - Y)'_{-v}(W^s), \right. \right. \\ & \left. \left. g_n(v) \right\rangle_{F \rightarrow F}, \gamma_n(y) \right\rangle_{F \rightarrow F} dv dy \\ & + \int_F \sum_{k \in \mathbb{Z} \setminus \{0\}} \left\langle \left\langle \Delta(A - Y)_{\tau_k}(W^s + (x + y)\mathbb{I}) - \Delta(A - Y)_{\tau_k}(W^s), \right. \right. \\ & \left. \left. g_n(\tau_{k'} - s) \right\rangle_{F \rightarrow F}, \gamma_n(y) \right\rangle_{F \rightarrow F} dy, \quad x \in F. \end{aligned}$$

By similar arguments as in the previous step, using conditions (1)-(3) of Section 1, (3.43) and this representation of  $K$  yield

$$\begin{aligned} |\nabla_{W_0}\psi(s)| \leq & \alpha'(S; W) \int_{r=0}^s |\nabla_{W_0}\psi(r)| dr \\ & + \beta'(S; W) \sum_{k' \in \mathbb{Z} \setminus \{0\}} \int_{r=0}^s |\nabla_{W_0}\psi(r)| \cdot |g_n(\tau_{k'} - r)| dr + |\kappa(s)| \end{aligned} \quad (3.44)$$

for  $s \in S$ ,  $n \in \mathbb{N}$ , and any finite subinterval  $S \subset \mathbb{R}$ . With

$$\delta(r; n, W) := \alpha'(S; W) + \beta'(S; W) \sum_{k' \in \mathbb{Z} \setminus \{0\}} |g_n(\tau_{k'} - r)|$$

we obtain as in (3.36) from (3.44) and Gronwall's inequality

$$\begin{aligned} & \left| \nabla_{W_0} (A_{n,s} - Y_{n,s}) - (\nabla_{W_0} A_s - \nabla_{W_0} Y_s) \right| = |\nabla_{W_0} \psi(s)| \\ & \leq |\kappa(s; n, W)| + \exp \left\{ \int_{r=0}^s \delta(r; n, W) dr \right\} \cdot \int_{r=0}^s |\kappa(r; n, W)| \cdot \delta(r; n, W) dr, \end{aligned} \quad (3.45)$$

$n \in \mathbb{N}$ ,  $s \in S$  for any finite, without loss of generality non-negative, subinterval  $S \subset \mathbb{R}$ . Adjusting the arguments of (3.34) and (3.35) to  $\nabla_{W_0} A$  and  $\nabla_{W_0} Y$  and in particular noticing the right continuity of  $\nabla_{W_0} A$  and  $\nabla_{W_0} Y$ , cf. Remark (7) of Section 1, we get from (3.39)

$$\kappa(s; n, W) \xrightarrow{n \rightarrow \infty} 0$$

$$|\kappa(\cdot; n, W)| \cdot \delta(\cdot; n, W) \xrightarrow{n \rightarrow \infty} 0 \quad \text{in } L^1([0, s])$$

if, without loss of generality,  $0 \leq s$  and neither 0 nor  $s$  is a jump time for  $X$ . Together with (3.45) we get from here that

$$\begin{aligned} & \nabla_{W_0} ((A_{n,l} - Y_{n,l}) - (A_{n,r} - Y_{n,r})) \\ & \xrightarrow{n \rightarrow \infty} (\nabla_{W_0} A_l - \nabla_{W_0} Y_l) - (\nabla_{W_0} A_r - \nabla_{W_0} Y_r), \quad l < r, \end{aligned}$$

if neither  $l$  nor  $r$  is a jump time for  $X$ .

Now we take special attention to condition (1) (iv), the uniform boundedness of  $\nabla_{W_0} A'$  in condition (2), the pendant for  $\nabla_{W_0} Y'$  according to condition (3) in Section 1, and Remark (7) of Section 1. Reviewing the definitions of  $\delta$  and  $\kappa$  via the definitions of  $\alpha'$ ,  $\beta'$ , and (3.34) we verify that the latter limit even holds uniformly bounded  $Q_\nu^{(m)}$ -a.e. on  $\{\pi_m W : W \in \Omega\}$ . We have shown (c) where, for its formulation, we recall that according to Remark (3) of this section  $\nabla_{W_0} A$  and  $\nabla_{W_0} Y$  jump only on  $Q_\nu^{(m)}$ -zero sets.  $\square$

**Proof of part (b).** *Step 1* In this step we verify the claims relative to  $A - Y$  and  $\dot{A} - \dot{Y}$ . We apply Gronwall's inequality for the proofs of the limits.

Let  $W \in \{\pi_m V : V \in \Omega\}$  with  $W_0 \notin G(W - W_0 \mathbb{1})$  and  $W^{(r)} := \pi_{m,r} W$ . As above, let  $\varphi$  denote the solution to (3.14) but let  $\varphi^{(r)}$  denote the solution to (3.14) for  $W$  replaced with  $W^{(r)}$ . Let  $s \in \mathbb{R}$ ,  $\varepsilon > 0$ , and choose  $r \in \mathbb{N}$  such that  $|s| + 1 < rt$  and

$$\begin{aligned} & \int_{v \in (-\frac{1}{n}, \frac{1}{n})} \left| A_{-v}(\cdot, \gamma_n) (W_{\cdot+s} + \varphi^{(r)}(s) \mathbb{1}) - Y_{-v}(\cdot, \gamma_n) (W_{\cdot+s} + \varphi^{(r)}(s) \mathbb{1}) \right. \\ & \quad \left. - \left( A_{-v}(\cdot, \gamma_n) (W_{\cdot+s}^{(r)} + \varphi^{(r)}(s) \mathbb{1}) - Y_{-v}(\cdot, \gamma_n) (W_{\cdot+s}^{(r)} + \varphi^{(r)}(s) \mathbb{1}) \right) \right| dv \\ & < \frac{n\varepsilon}{2\|g'_n\|}. \end{aligned} \quad (3.46)$$

Such a choice of  $r \in \mathbb{N}$  is possible according to condition (4) (ii) of Section 1. It follows from Remark (6) of Section 1 that there is some  $\beta > 0$  not depending on  $r$  such that

$$\begin{aligned} & \left| A_{-v}(\cdot, \gamma_n) (W_{\cdot+s} + \varphi(s) \mathbb{1}) - A_{-v}(\cdot, \gamma_n) (W_{\cdot+s} + \varphi^{(r)}(s) \mathbb{1}) \right. \\ & \quad \left. - \left( Y_{-v}(\cdot, \gamma_n) (W_{\cdot+s} + \varphi(s) \mathbb{1}) - Y_{-v}(\cdot, \gamma_n) (W_{\cdot+s} + \varphi^{(r)}(s) \mathbb{1}) \right) \right| \\ & < \frac{n\beta}{2\|g'_n\|} \cdot |\varphi(s) - \varphi^{(r)}(s)| \end{aligned} \quad (3.47)$$

for all  $v \in (-\frac{1}{n}, \frac{1}{n})$ ,  $|s| + 1 < rt$ . It follows from (3.14) and (3.15) that

$$\begin{aligned} \dot{\varphi}(s) - \dot{\varphi}^{(r)}(s) &= \int_{(-\frac{1}{n}, \frac{1}{n})} \left\langle A_{-v}(\cdot, \gamma_n) (W_{\cdot+s} + \varphi(s) \mathbb{1}) \right. \\ &\quad \left. - A_{-v}(\cdot, \gamma_n) \left( W_{\cdot+s}^{(r)} + \varphi^{(r)}(s) \mathbb{1} \right), g'_n(v) \right\rangle_{F \rightarrow F} dv \\ &\quad - \int_{(-\frac{1}{n}, \frac{1}{n})} \left\langle Y_{-v}(\cdot, \gamma_n) (W_{\cdot+s} + \varphi(s) \mathbb{1}) \right. \\ &\quad \left. - Y_{-v}(\cdot, \gamma_n) \left( W_{\cdot+s}^{(r)} + \varphi^{(r)}(s) \mathbb{1} \right), g'_n(v) \right\rangle_{F \rightarrow F} dv. \end{aligned}$$

Together with relation (3.46) and (3.47), this implies

$$\left| \dot{\varphi}(s) - \dot{\varphi}^{(r)}(s) \right| < \varepsilon + \beta \cdot \left| \varphi(s) - \varphi^{(r)}(s) \right|. \quad (3.48)$$

By (3.14) we have  $\varphi(0) = 0 = \varphi^{(r)}(0)$ . Moreover, because of (3.18) and (3.23) it holds that  $\varphi(s) = A_{n,s}(W) - Y_{n,s}(W)$  and  $\varphi^{(r)}(s) = A_{n,s}(W^{(r)}) - Y_{n,s}(W^{(r)})$ . By means of Gronwall's inequality, (3.48) implies now

$$\begin{aligned} \left| \dot{A}_{n,s}(W) - \dot{Y}_{n,s}(W) - \left( \dot{A}_{n,s}(W^{(r)}) - \dot{Y}_{n,s}(W^{(r)}) \right) \right| &= \left| \dot{\varphi}(s) - \dot{\varphi}^{(r)}(s) \right| \\ &\leq \varepsilon \cdot e^{\beta|s|}, \quad |s| \leq rt. \end{aligned} \quad (3.49)$$

We have thus verified  $\dot{A}_{n,\cdot}(\pi_{m,r}W) - \dot{Y}_{n,\cdot}(\pi_{m,r}W) \xrightarrow{r \rightarrow \infty} \dot{A}_{n,\cdot}(W) - \dot{Y}_{n,\cdot}(W)$ , uniformly on every finite subinterval  $S \subset \mathbb{R}$ . Relation  $A_{n,\cdot}(\pi_{m,r}W) - Y_{n,\cdot}(\pi_{m,r}W) \xrightarrow{r \rightarrow \infty} A_{n,\cdot}(W) - Y_{n,\cdot}(W)$ , uniformly on every finite subinterval  $S \subset \mathbb{R}$  follows now from (3.49) together with  $A_{n,0} - Y_{n,0} = 0$ , cf. (3.22). The latter also implies

$$(W_v^{(r)})^{n,s} \xrightarrow{r \rightarrow \infty} W_v^{n,s} \quad (3.50)$$

uniformly on  $(s, v) \in S^2$  for every finite subinterval  $S \subset \mathbb{R}$ . Furthermore, taking the absolute value  $|\cdot|$  and the  $\sup_{r \in \mathbb{N}}$  individually for each coordinate of  $F$ ,

$$\sup_{r \in \mathbb{N}} \left| \dot{A}_{n,s}(\pi_{m,r}W) - \dot{Y}_{n,s}(\pi_{m,r}W) \right| \in \bigcap_{1 \leq p < \infty} L^p(\Omega, Q_\nu^{(m)}; F), \quad s \in \mathbb{R}, \quad (3.51)$$

is a consequence of (3.38) and the finite Lévy-Ciesielsky construction of  $\pi_{m,r}W$  in (2.1). Note that the  $\langle H_i, dW \rangle_{L^2}$ ,  $i \in I(m, r)$ , in (2.1) are independent  $N(0, 1)$ -random variables and that by the  $\sup_{v \in (-\frac{1}{n}, \frac{1}{n})}$  in (3.38) we use just a selection of them which is independent of  $r$ .

*Step 2* We prove the claims relative to  $\nabla_{W_0} \dot{A} - \nabla_{W_0} \dot{Y}$ . Again we use Gronwall's inequality.

Let  $W \in \{\pi_m V : V \in \Omega\}$  with  $W_0 \notin G(W - W_0 \mathbb{1})$ . Recalling (3.22), (3.23), as well as (3.40) and (3.41) we obtain

$$\begin{aligned} \left( \nabla_{W_0} A_{n,s} \right)'(W^{(r)}) - \left( \nabla_{W_0} Y_{n,s} \right)'(W^{(r)}) &= \nabla_{W_0} \dot{A}_{n,s}(W^{(r)}) - \nabla_{W_0} \dot{Y}_{n,s}(W^{(r)}) \\ &= \nabla_{W_0} \left( \dot{C}_{n,0} \circ (W^{(r)})^{n,s} \right) \\ &= \left( \nabla_{W_0} \dot{C}_{n,0} \right) \left( (W^{(r)})^{n,s} \right) \cdot \left( \nabla_{W_0} A_{n,s}(W^{(r)}) - \nabla_{W_0} Y_{n,s}(W^{(r)}) + e \right), \end{aligned} \quad (3.52)$$

$s \in \mathbb{R}$ . Let us now consider (3.52) a first order system

$$\begin{aligned}\dot{\xi}(s) &= \left( \nabla_{W_0} \dot{C}_{n,0} \right) \left( (W^{(r)})^{n,s} \right) \cdot (\xi(s) + \mathbf{e}) , \quad s \in \mathbb{R}, \\ \xi(0) &= 0\end{aligned}\tag{3.53}$$

with solution

$$\xi(s) = \nabla_{W_0} A_{n,s} (W^{(r)}) - \nabla_{W_0} Y_{n,s} (W^{(r)}) , \quad |s| \leq rt,$$

which is unique by the arguments for (3.40)-(3.42). Together with representations (3.41), (3.42), relation (3.50), and condition (4) (i) of Section 1 we get

$$\left( \nabla_{W_0} \dot{C}_{n,0} \right) \left( (W^{(r)})^{n,s} \right) \xrightarrow{r \rightarrow \infty} \left( \nabla_{W_0} \dot{C}_{n,0} \right) (W^{n,s})$$

uniformly on every finite subinterval  $s \in S \subset \mathbb{R}$ . Now it follows from (3.40) and (3.53) that

$$\nabla_{W_0} \dot{A}_{n,s} (W^{(r)}) - \nabla_{W_0} \dot{Y}_{n,s} (W^{(r)}) \xrightarrow{r \rightarrow \infty} \nabla_{W_0} \dot{A}_{n,s}(W) - \nabla_{W_0} \dot{Y}_{n,s}(W) , \quad s \in \mathbb{R}.\tag{3.54}$$

By (3.40)-(3.42) we have

$$\begin{aligned}& \nabla_{W_0} \dot{A}_{n,s}(W) - \nabla_{W_0} \dot{Y}_{n,s}(W) \\ &= \left( \int_{(-\frac{1}{n}, \frac{1}{n})} \left\langle \int_F \left\langle \left( A_{-v}(\cdot + x\mathbb{I}) - Y_{-v}(\cdot + x\mathbb{I}) \right) \mathbf{e}, \right. \right. \\ & \quad \left. \left. \nabla_{W_0} \gamma_n(x) \right\rangle_{F \rightarrow F} dx, g'_n(v) \right\rangle_{F \rightarrow F} dv \right) (W^{n,s}) \times \\ & \quad \times \left( \nabla_{W_0} A_{n,s}(W) - \nabla_{W_0} Y_{n,s}(W) + \mathbf{e} \right) , \quad s \in \mathbb{R}.\end{aligned}\tag{3.55}$$

Let us set

$$b(n) := \frac{2 \|g'_n\| \|\nabla \gamma_n\| \lambda_F \left( B_{\frac{1}{n^3}} \right)}{n}$$

where  $B_{\frac{1}{n^3}} \subset F$  denotes any ball with radius  $\frac{1}{n^3}$ . As for the verification of (3.38) we obtain from (3.55)

$$\begin{aligned}& \left| \nabla_{W_0} \dot{A}_{n,s}(W) - \nabla_{W_0} \dot{Y}_{n,s}(W) \right| \\ & \leq \left( a(n) + b(n) \sup_{v \in (-\frac{1}{n}, \frac{1}{n})} |W_{-v+s} - W_s| \right) \times \\ & \quad \times \left( 1 + \left| \nabla_{W_0} A_{n,s}(W) - \nabla_{W_0} Y_{n,s}(W) \right| \right)\end{aligned}\tag{3.56}$$

where  $a(n)$  is some positive constant which may depend on  $n \in \mathbb{N}$  but is independent of  $s \in \mathbb{R}$  and  $W \in \{\pi_m V : V \in \Omega\}$ . The absolute values on the left-hand side and in the second

factor of the right-hand side have been taken coordinate wise. Taking into consideration  $\nabla_{W_0} A_{n,0} - \nabla_{W_0} Y_{n,0} = 0$ , Gronwall's inequality shows now that

$$\begin{aligned} \left| \nabla_{W_0} A_{n,s}(W) - \nabla_{W_0} Y_{n,s}(W) \right| &\leq s \cdot \left( a(n) + 2b(n) \sup_{u \in (-\frac{1}{n}, s + \frac{1}{n})} |W_u - W_0| \right) \times \\ &\times \int_{u=0}^s \exp \left\{ a(n) + b(n) \sup_{v \in (-\frac{1}{n}, \frac{1}{n})} |W_{-v+u} - W_u| \right\} du \end{aligned} \quad (3.57)$$

for, without loss of generality,  $s \geq 0$ . In order to verify that all exponential moments of  $\sup_{v \in (-\frac{1}{n}, \frac{1}{n})} c \cdot |W_{-v+u} - W_u|$ ,  $c > 0$ , are finite with respect to the measure  $Q_\nu^{(m)}$  and bounded with respect to  $u \in [0, s]$  one may read, for example [15], V.8, the proof of the lemma following Theorem 43 therein, and adjust this to the measure  $Q_\nu^{(m)}$ . Applying the arguments of (3.51) to the composition of (3.56) with (3.57) we obtain

$$\sup_{r \in \mathbb{N}} \left| \nabla_{W_0} \dot{A}_{n,s}(\pi_{m,r} W) - \nabla_{W_0} \dot{Y}_{n,s}(\pi_{m,r} W) \right| \in L^1(\Omega, Q_\nu^{(m)}; F), \quad s \in \mathbb{R}, \quad (3.58)$$

the absolute value  $|\cdot|$  and the  $\sup_{r \in \mathbb{N}}$  has been taken coordinate wise.

Everything stated in (b) is now in (3.49) as well as the sentence underneath, (3.51), (3.54), and (3.58).  $\square$

**Proof of the remaining parts (ii)-(iv).** Item (iv) of the lemma is a direct consequence of (3.56) together with (3.57). As an immediate consequence of (3.38) we get

$$\dot{A}_{n,s} - \dot{Y}_{n,s} \in \bigcap_{1 \leq p < \infty} L^p(\Omega, Q_\nu^{(m,r)}; F) \cap L^p(\Omega, Q_\nu^{(m)}; F), \quad s \in \mathbb{R},$$

and verify that  $\dot{A}_n(W) - \dot{Y}_n(W)$  is bounded on finite subintervals of  $\mathbb{R}$  for all  $W \in \{\pi_m V : V \in \Omega\}$ . In other words, we have (iii) of the present lemma.

The remainder of the proof focuses on (ii). In the first two steps we prove the statements about  $\nabla_H X$  and  $\nabla_H Y$ .

*Step 1* In this step we construct path wise estimates on the individual directional derivatives of  $A_{n,s}$  and  $Y_{n,s}$  with respect to coordinates  $\kappa_i$  and  $\lambda_i$ . Furthermore, we prove belongingness of such derivatives to the stated  $L^p$ -spaces.

According to (3.12), (3.13), and (3.20)-(3.22) we have

$$A_{n,s}(W) - Y_{n,s}(W) = \int_0^s \left( A_0(g'_n, \gamma_n)(W^{n,u}) - Y_0(g'_n, \gamma_n)(W^{n,u}) \right) du \quad (3.59)$$

and

$$Y_{n,s}(W) = A_0(g_n, \gamma_n)(W^{n,s}), \quad s \in \mathbb{R}. \quad (3.60)$$

In the following calculations we will use the notation  $\lambda_j = e_j \mathbb{I}(\cdot)$ , for all coordinates  $j \in F$ , here  $F = \mathbb{R}^{n-d}$ . In order to simplify the notation, let us write  $(A - Y)_0(g'_n, \gamma_n)$  instead of  $A_0(g'_n, \gamma_n) - Y_0(g'_n, \gamma_n)$ . According to condition (5) (ii) of Section 1 the equation

$$\begin{aligned} Z_{i,s} &= \int_0^s \left\langle \left( \nabla_H (A - Y)_0(g'_n, \gamma_n) \right) (W^{n,u}), (H_i(\cdot + u), Z_{i,u}) \right\rangle_{H \rightarrow F} du \\ Z_{i,0} &= 0 \end{aligned} \quad (3.61)$$

is well-defined on  $jH$ . By the continuity of  $u \rightarrow W^{n,u}$  in the sense of (3.19) and the Picard-Lindelöf theorem there exists a unique solution in some neighborhood  $U$  of  $s = 0$ . It follows also from the Picard-Lindelöf theorem that  $U$  can be chosen so that it is independent of  $i \in I(r)$ . Recalling again condition (5) (ii) of Section 1 we notice that

$$\begin{aligned} & \left\langle \left( \nabla_H (A - Y)_0 (g'_n, \gamma_n) \right) (W^{n,u}), (H_i(\cdot + u), Z_{i,u}) \right\rangle_{H \rightarrow F} \\ &= \sum_{i' \in I(r)} \left( \frac{\partial}{\partial \kappa_{i'}} \dot{C}_{n,0} \right) (W^{n,u}) \cdot \frac{\partial}{\partial \kappa_i} \langle H_{i'}, dW^{n,u} \rangle_{L^2} \\ & \quad + \sum_{j'} \left( \frac{\partial}{\partial \lambda_{j'}} \dot{C}_{n,0} \right) (W^{n,u}) \cdot \langle e_{j'}, Z_{i,u} \rangle_F. \end{aligned}$$

Now we obtain from the facts that  $W^{n,\cdot}$  is the unique solution to (3.19) and that condition (5) (ii) provides the chain rule the existence of the derivative  $\frac{\partial}{\partial \kappa_i} (A_{n,s} - Y_{n,s})$  and

$$Z_{i,s} = \frac{\partial}{\partial \kappa_i} (A_{n,s} - Y_{n,s})$$

for all  $s$  in some neighborhood  $U$  of 0. This implies the existence of the derivative  $\frac{\partial}{\partial \kappa_i} Y_{n,s}$  for all  $s$  in the same neighborhood  $U$  of 0 and

$$\frac{\partial}{\partial \kappa_i} Y_{n,s} = \left\langle \left( \nabla_H A_0 (g_n, \gamma_n) \right) (W^{n,s}), \left( H_i(\cdot + s), \frac{\partial}{\partial \kappa_i} (A_{n,s} - Y_{n,s}) \right) \right\rangle_{H \rightarrow F}. \quad (3.62)$$

Let  $s \in \mathcal{R} \cap U \equiv \mathcal{R}(r) \cap U$  and, without loss of generality,  $s \geq 0$ . With condition (5) (i) of Section 1 it follows from (3.61) that

$$\begin{aligned} \left| \frac{\partial}{\partial \kappa_i} A_{n,s} - \frac{\partial}{\partial \kappa_i} Y_{n,s} \right| &= \left| \int_0^s \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left( \left( (A - Y)_0 (g'_n, \gamma_n) \right) (W^{n,u} + \varepsilon \cdot j(H_i(\cdot + u), Z_{i,u})) \right. \right. \\ & \quad \left. \left. - \left( (A - Y)_0 (g'_n, \gamma_n) \right) (W^{n,u}) \right) du \right| \\ &\leq \int_0^s \left( \left| \left( \nabla_{W_0} A_0 (|g'_n|, \gamma_n) \right) (W^{n,u}) \right| + \left| \left( \nabla_{W_0} Y_0 (|g'_n|, \gamma_n) \right) (W^{n,u}) \right| \right) \times \\ & \quad \times \left( |\langle H_i(\cdot + u), h_u \rangle_{L^2}| + \left| \frac{\partial}{\partial \kappa_i} A_{n,u} - \frac{\partial}{\partial \kappa_i} Y_{n,u} \right| \right) du \end{aligned} \quad (3.63)$$

and from (3.62) that

$$\begin{aligned} \left| \frac{\partial}{\partial \kappa_i} Y_{n,s} \right| &\leq \left| \left( \nabla_{W_0} A_0 (g_n, \gamma_n) \right) (W^{n,s}) \right| \times \\ & \quad \times \left( |\langle H_i(\cdot + s), h_s \rangle_{L^2}| + \left| \frac{\partial}{\partial \kappa_i} A_{n,s} - \frac{\partial}{\partial \kappa_i} Y_{n,s} \right| \right) \end{aligned} \quad (3.64)$$

where  $h_\cdot \in H$  is defined in condition (5) (i) of Section 1.

For  $g = |g'_n|$  or  $g = g_n$  and  $W \in jH$  we obtain as in (3.38) or (3.55), (3.56)

$$\left| \left( \nabla_{W_0} A_0 (g, \gamma_n) \right) (W^{n,u}) \right| = \left| \left( \int_{(-\frac{1}{n}, \frac{1}{n})} \left\langle \int_F \langle A_{-v}(\cdot + (x - W_0) \mathbf{1}) \mathbf{e}, \right. \right. \right.$$

$$\begin{aligned} & \left| \left\langle \nabla_{W_0} \gamma_n (x - W_0) \right\rangle_{F \rightarrow F} dx, g(v) \right\rangle_{F \rightarrow F} dv \right| (W^{n,u}) \Big| \\ & \leq d(n) \left( 1 + \sup_{v \in (-\frac{1}{n}, \frac{1}{n})} |W_{-v+u} - W_u| \right) \end{aligned} \quad (3.65)$$

as well as

$$\left| \left( \nabla_{W_0} Y_0 (g, \gamma_n) \right) (W^{n,u}) \right| \leq d(n), \quad (3.66)$$

$u \in [0, s]$ , where  $d(n)$  is some positive constant which may depend on  $n \in \mathbb{N}$  but is independent of  $u \in [0, s]$  and  $W \in jH$ . Relations (3.63), (3.65), and (3.66) yield

$$\begin{aligned} \left| \frac{\partial}{\partial \kappa_i} A_{n,s} - \frac{\partial}{\partial \kappa_i} Y_{n,s} \right| & \leq \int_0^s d(n) \left( 2 + \sup_{v \in (-\frac{1}{n}, \frac{1}{n})} |W_{-v+u} - W_u| \right) \times \\ & \times \left( |\langle H_i(\cdot + u), h_u \rangle_{L^2}| + \left| \frac{\partial}{\partial \kappa_i} A_{n,u} - \frac{\partial}{\partial \kappa_i} Y_{n,u} \right| \right) du. \end{aligned} \quad (3.67)$$

Recalling (3.61) we verify that  $u \rightarrow \frac{\partial}{\partial \kappa_i} A_{n,u} - \frac{\partial}{\partial \kappa_i} Y_{n,u}$  is a continuous function on  $u \in [0, s]$ . Applying now Gronwall's inequality, relation (3.67) shows that with

$$c(W) := 2d(n) \sup_{u \in (-\frac{1}{n}, s + \frac{1}{n})} \left( 1 + |W_u - W_0| \right) \quad (3.68)$$

we have

$$\begin{aligned} \left| \frac{\partial}{\partial \kappa_i} A_{n,s} - \frac{\partial}{\partial \kappa_i} Y_{n,s} \right| & \leq c(W) \int_0^s |\langle H_i(\cdot + u), h_u \rangle_{L^2}| du \\ & + c(W) \cdot e^{s \cdot c(W)} \int_0^s \int_0^u |\langle H_i(\cdot + v), h_v \rangle_{L^2}| dv du. \end{aligned} \quad (3.69)$$

Together with (3.64) and (3.65) we obtain from (3.69)

$$\begin{aligned} \left| \frac{\partial}{\partial \kappa_i} Y_{n,s}(W) \right| & \leq c(W) \cdot |\langle H_i(\cdot + s), h_s \rangle_{L^2}| + (c(W))^2 \int_0^s |\langle H_i(\cdot + u), h_u \rangle_{L^2}| du \\ & + (c(W))^2 \cdot e^{s \cdot c(W)} \int_0^s \int_0^u |\langle H_i(\cdot + v), h_v \rangle_{L^2}| dv du \end{aligned} \quad (3.70)$$

and

$$\begin{aligned} \left| \frac{\partial}{\partial \kappa_i} A_{n,s}(W) \right| & \leq c(W) \cdot |\langle H_i(\cdot + s), h_s \rangle_{L^2}| + (c(W) + (c(W))^2) \int_0^s |\langle H_i(\cdot + u), h_u \rangle_{L^2}| du \\ & + (c(W) + (c(W))^2) \cdot e^{s \cdot c(W)} \int_0^s \int_0^u |\langle H_i(\cdot + v), h_v \rangle_{L^2}| dv du \end{aligned} \quad (3.71)$$

for all  $W \in jH$  and all  $i \in I(r)$ . Recalling the argument after (3.57) this gives

$$\frac{\partial}{\partial \kappa_i} A_{n,s}, \frac{\partial}{\partial \kappa_i} Y_{n,s} \in \bigcap_{1 \leq p < \infty} L^p(\Omega, Q_\nu^{(m,r)}; F), \quad i \in I(r). \quad (3.72)$$

Similarly, we get

$$\frac{\partial}{\partial \lambda_j} A_{n,s}, \frac{\partial}{\partial \lambda_j} Y_{n,s} \in \bigcap_{1 \leq p < \infty} L^p(\Omega, Q_\nu^{(m,r)}; F) \quad (3.73)$$

by examining

$$\begin{aligned} z_{j,s} &= \int_0^s \left\langle \left( \nabla_H ((A - Y)_0 (g'_n, \gamma_n)) \right) (W^{n,u}), (0, e_j + z_{j,u}) \right\rangle_{H \rightarrow F} du \\ z_{j,0} &= 0 \end{aligned} \quad (3.74)$$

and

$$\frac{\partial}{\partial \lambda_j} Y_{n,s} = \left\langle \left( \nabla_H A_0 (g_n, \gamma_n) \right) (W^{n,s}), \left( 0, e_j + \frac{\partial}{\partial \lambda_j} (A_{n,s} - Y_{n,s}) \right) \right\rangle_{H \rightarrow F}. \quad (3.75)$$

Indeed, as above, it follows that  $z_{j,s} = \frac{\partial}{\partial \lambda_j} (A_{n,s} - Y_{n,s})$  is the unique solution to (3.74) for all  $s$  in some neighborhood of 0.

*Step 2* In this step we represent  $\nabla_H X_{n,s}$  and  $\nabla_H Y_{n,s}$  by means of the directional derivatives and prove belongingness to the stated  $L^p$ -spaces. As a byproduct of Steps 1 and 2 we verify that  $A_{n,s}$  and  $Y_{n,s}$  belong to the stated  $D_{q,1}$ -spaces.

Let

$$\begin{aligned} k &= \sum_{i \in I(r)} \langle k, (H_i, 0) \rangle_H \cdot (H_i, 0) + \sum_j \langle k, (0, e_j) \rangle_H \cdot (0, e_j) \\ &=: k^{(1)} + k^{(2)} \equiv (k^{(1)}, k^{(2)}) \in H \end{aligned}$$

and

$$\kappa^{(1)} := \int_0^\cdot k^{(1)}(v) dv, \quad \kappa^{(2)} := k^{(2)} \mathbb{I}(\cdot), \quad \kappa := \kappa^{(1)} + \kappa^{(2)}.$$

On the one hand, keeping condition (5) (ii) in mind, as in (3.61) and (3.62) it turns out that the derivative  $\frac{\partial}{\partial \kappa} (A_{n,s} - Y_{n,s})$  exists on  $jH$ . In fact,

$$Z_s := \frac{\partial}{\partial \kappa} (A_{n,s} - Y_{n,s})$$

is the unique solution to

$$\begin{aligned} Z_s &= \int_0^s \left\langle \left( \nabla_H ((A - Y)_0 (g'_n, \gamma_n)) \right) (W^{n,u}), (k^{(1)}(\cdot + u), k^{(2)} + Z_u) \right\rangle_{H \rightarrow F} du \\ Z_0 &= 0 \end{aligned} \quad (3.76)$$

for all  $s$  in some neighborhood of 0 and

$$\begin{aligned} \frac{\partial}{\partial \kappa} Y_{n,s} &= \left\langle \left( \nabla_H A_0 (g_n, \gamma_n) \right) (W^{n,s}), \right. \\ &\quad \left. \left( k^{(1)}(\cdot + s), k^{(2)} + \frac{\partial}{\partial \kappa} (A_{n,s} - Y_{n,s}) \right) \right\rangle_{H \rightarrow F}. \end{aligned} \quad (3.77)$$

On the other hand, by (3.70) we get

$$\begin{aligned} \left( \frac{\partial}{\partial \kappa_i} Y_{n,s}(W) \right)^2 &\leq 3(c(W))^2 \cdot \langle H_i(\cdot + s), h_s \rangle_{L^2}^2 + 3(c(W))^4 rt \int_0^s \langle H_i(\cdot + u), h_u \rangle_{L^2}^2 du \\ &\quad + 3(c(W))^4 \cdot e^{2s \cdot c(W)} (rt)^2 \int_0^s \int_0^u \langle H_i(\cdot + v), h_v \rangle_{L^2}^2 dv du \end{aligned}$$

where we recall the definition of  $c(W)$  in (3.68). With  $h_s$  extending to a function of type  $\mathbb{R} \rightarrow F$  by setting  $h_s = 0$  on  $\mathbb{R} \setminus \mathcal{R}$  and using the abbreviation  $c \equiv c(W)$  the latter implies

$$\begin{aligned} \sum_{i \in I(r)} \left( \frac{\partial}{\partial \kappa_i} Y_{n,s}(W) \right)^2 &\leq 3(c(W))^2 \cdot \|h_s(\cdot - s)\|_{L^2}^2 + 3(c(W))^4 rt \int_0^s \|h_u(\cdot - u)\|_{L^2}^2 du \\ &\quad + 3(c(W))^4 \cdot e^{2s \cdot c(W)} (rt)^2 \int_0^s \int_0^u \|h_v(\cdot - v)\|_{L^2}^2 dv du \\ &\leq \left( 3c^2 + 3c^4 ((rt)^2 + (rt)^4 e^{2rt \cdot c}) \right) \cdot \sup_{s \in \mathcal{R}} \|h_s\|^2. \end{aligned} \quad (3.78)$$

Similarly, by (3.71)

$$\sum_{i \in I(r)} \left( \frac{\partial}{\partial \kappa_i} A_{n,s}(W) \right)^2 \leq \left( 3c^2 + 3(c + c^2)^2 ((rt)^2 + (rt)^4 e^{2rt \cdot c}) \right) \cdot \sup_{s \in \mathcal{R}} \|h_s\|^2. \quad (3.79)$$

In other words, we obtain by linear combination of (3.61) and (3.62) as well as (3.74) and (3.75) and from the uniqueness of (3.76) and (3.77)

$$\frac{\partial}{\partial \kappa} A_{n,s} = \sum_{i \in I(r)} \langle k, (H_i, 0) \rangle_H \cdot \frac{\partial}{\partial \kappa_i} A_{n,s} + \sum_j \langle k, (0, e_j) \rangle_H \cdot \frac{\partial}{\partial \lambda_j} A_{n,s} \quad (3.80)$$

and

$$\frac{\partial}{\partial \kappa} Y_{n,s} = \sum_{i \in I(r)} \langle k, (H_i, 0) \rangle_H \cdot \frac{\partial}{\partial \kappa_i} Y_{n,s} + \sum_j \langle k, (0, e_j) \rangle_H \cdot \frac{\partial}{\partial \lambda_j} Y_{n,s} \quad (3.81)$$

in some neighborhood of  $s = 0$  on  $jH$ . Now we replace in (3.61), (3.74), and (3.76) the argument  $W^{n,u}$  by  $W^{n,u+s_0}$  as well as in (3.62), (3.75), and (3.77)  $W^{n,s}$  by  $W^{n,s+s_0}$ . By adjusting the initial conditions in (3.61), (3.74), and (3.76) we verify (3.80) and (3.81) for all  $s = s_0 \in \mathbb{R}$ . Together with (3.78), (3.79) and  $c \equiv c(W)$  in (3.68) this says

$$\nabla_H A_{n,s} = \sum_{i \in I(r)} \frac{\partial}{\partial \kappa_i} A_{n,s} \cdot (H_i, 0) + \sum_j \frac{\partial}{\partial \lambda_j} A_{n,s} \cdot (0, e_j), \quad (3.82)$$

$$\nabla_H Y_{n,s} = \sum_{i \in I(r)} \frac{\partial}{\partial \kappa_i} Y_{n,s} \cdot (H_i, 0) + \sum_j \frac{\partial}{\partial \lambda_j} Y_{n,s} \cdot (0, e_j), \quad s \in \mathbb{R}, \quad (3.83)$$

and

$$\nabla_H X_{n,s} \in \bigcap_{1 \leq p < \infty} L^p(\Omega, Q_\nu^{(m,r)}; H) \quad \text{and} \quad \nabla_H Y_{n,s} \in \bigcap_{1 \leq p < \infty} L^p(\Omega, Q_\nu^{(m,r)}; H).$$

We recall (3.59) and (3.60), as well as the paragraph below (3.37) in order to verify  $A_{n,s}, Y_{n,s} \in \bigcap_{1 \leq p < \infty} L^p(\Omega, Q_\nu^{(m,r)}; F)$ . It follows now from (3.72), (3.73) and (3.81) and Theorem 1.11 in [9] that  $A_{n,s}, Y_{n,s} \in D_{q,1}(Q_\nu^{(m,r)})$  for all  $q$  with  $1/q + 1/Q < 1$ .

*Step 3* It remains to verify property (iii) of Proposition 2.7 for  $X_n$  as well as  $Y_n$ . Let us take a look at (3.82) and (3.83). It turns out that

$$D_G A_{n,s}(W) = \sum_{i \in I(r)} \frac{\partial}{\partial \kappa_i} A_{n,s}(W) \cdot (H_i, 0) + \sum_j \frac{\partial}{\partial \lambda_j} A_{n,s}(W) \cdot (0, e_j)$$

is the Gâteaux derivative in the space  $jH \equiv jH^{(r)}$  of the function  $A_{n,s}$ ,  $n \in \mathbb{N}$ ,  $s \in \mathbb{R}$ , at the point  $W \in jH$ . Similarly we get the Gâteaux derivative of  $Y_{n,s}$ .

We observe that all  $\frac{\partial}{\partial \kappa_i} A_{n,s}$ ,  $\frac{\partial}{\partial \kappa_i} Y_{n,s}$ ,  $i \in I(r)$ , and all  $\frac{\partial}{\partial \lambda_j} A_{n,s}$ ,  $\frac{\partial}{\partial \lambda_j} Y_{n,s}$ ,  $s \in \mathcal{R}$ , are continuous on  $jH$ , i.e., with respect to the coordinates  $\kappa_i$ ,  $i \in I(r)$ , and  $\lambda_j$ . In fact, we just use (3.61), (3.62) as well as (3.74), (3.75) together with conditions (4) (i) and (5) (i) of Section 1 where we recall from Subsection 1.2 the definitions of the jump times, the relation  $\nu(G(W)) = 0$ ,  $W \in \Omega$ , and the definitions of  $A(\cdot, \gamma_n)$  as well as  $Y(\cdot, \gamma_n)$ . Furthermore, we take into consideration the continuity  $jH^{(r)} \ni W \rightarrow W^{n,u}$ ,  $u \in [0, s]$ . The latter we derive from (3.13), (3.14), and (3.18) and again condition (4) (i) of Section 1.

With (3.78), (3.79), and (3.68) it follows now that  $D_G A_{n,s}$  and  $D_G Y_{n,s}$  are continuous in the space  $jH$ . We conclude that  $D_G A_{n,s}(W)$  and  $D_G Y_{n,s}(W)$  are at the same time the Fréchet derivatives  $D_F A_{n,s}(W)$  and  $D_F Y_{n,s}(W)$  of  $A_{n,s}$  and  $Y_{n,s}$  in the space  $jH$ .

Next we show differentiability of  $(D_F A_{n,s}(W))_1$  and  $(D_F Y_{n,s}(W))_1$  for  $W \in jH$  and  $s \in \mathcal{R}$ . According to condition (5) (iii),  $\nabla_G(A_0 - Y_0)(g'_n, \gamma_n)$  is continuously differentiable which we symbolize by  $(\dot{\nabla}_G)(A_0 - Y_0)(g'_n, \gamma_n)$ . For fixed  $r \in \mathcal{R}$  and variable  $s \in \mathcal{R}$ , we consider the first order ODE

$$\begin{aligned} \zeta_s(r) &= \int_0^s \left( \left( (\dot{\nabla}_G)_{r-u}(A_0 - Y_0)(g'_n, \gamma_n) \right) (W^{n,u}) \right. \\ &\quad \left. + \left\langle \left( \nabla_{W_0}(A_0 - Y_0)(g'_n, \gamma_n) \right) (W^{n,u}), \zeta_u(r) \right\rangle_F \right) du \\ \zeta_0(r) &= 0. \end{aligned}$$

Using the arguments for (3.61), this equation has a unique solution. Furthermore,

$$Z_s(r) := \int_0^r \zeta_s(p) dp \tag{3.84}$$

is the unique solution to

$$\begin{aligned} Z_s(r) &= \int_0^s \left( \left( (\nabla_G)_{r-u}(A_0 - Y_0)(g'_n, \gamma_n) \right) (W^{n,u}) \right. \\ &\quad \left. + \left\langle \left( \nabla_{W_0}(A_0 - Y_0)(g'_n, \gamma_n) \right) (W^{n,u}), Z_u(r) \right\rangle_F \right) du \\ Z_0 &= 0. \end{aligned} \tag{3.85}$$

For  $W \in jH$  we have

$$\left( (\nabla_G)_{-u}(A_0 - Y_0)(g'_n, \gamma_n) \right) (W^{n,u})$$

$$\begin{aligned}
& + \left\langle \left( \nabla_{W_0}(A_0 - Y_0)(g'_n, \gamma_n) \right) (W^{n,u}), D_{F,1}(A_{n,u} - Y_{n,u})(W) \right\rangle_F \\
& = \sum_{i \in I(r)} H_i \left\langle \left( \nabla_G(A_0 - Y_0)(g'_n, \gamma_n) \right) (W^{n,u}), H_i(\cdot + u) \right\rangle_{L^2} \\
& \quad + \left\langle \left( \nabla_{W_0}(A_0 - Y_0)(g'_n, \gamma_n) \right) (W^{n,u}), D_{F,1}(A_{n,u} - Y_{n,u})(W) \right\rangle_F \\
& = \frac{d}{du} D_{F,1}(A_{n,u} - Y_{n,u})(W)
\end{aligned}$$

where, for the last equality sign we have used (3.61) and condition (5) (ii) of Section 1. This and  $D_{F,1}(A_{n,0} - Y_{n,0})(W) = 0$  on  $\mathcal{R}$ , (3.85) imply  $Z_s = D_{F,1}(A_{n,s} - Y_{n,s})(W)$ . Relation (3.84) says now that  $D_{F,1}A_{n,s}(W) - D_{F,1}Y_{n,s}(W)$  is differentiable.

As a consequence of (3.62) and condition (5) (ii) of Section 1, for  $W \in jH$  it holds that

$$\begin{aligned}
D_{F,1}Y_{n,s} & = \sum_{i \in I(r)} H_i \left\langle \left( \nabla_G A_0(g_n, \gamma_n) \right) (W^{n,s}), H_i(\cdot + s) \right\rangle_{L^2} \\
& \quad + \left\langle \left( \nabla_{W_0} A_0(g_n, \gamma_n) \right) (W^{n,s}), D_{F,1}(A_{n,s} - Y_{n,s})(W) \right\rangle_F \\
& = \left( (\nabla_G)_{\cdot - s} A_0(g_n, \gamma_n) \right) (W^{n,s}) \\
& \quad + \left\langle \left( \nabla_{W_0} A_0(g_n, \gamma_n) \right) (W^{n,s}), D_{F,1}(A_{n,s} - Y_{n,s})(W) \right\rangle_F.
\end{aligned}$$

Differentiability of  $D_{F,1}Y_{n,s}(W)$ ,  $W \in jH$ , follows now from condition (5) (iii) of Section 1 and differentiability of  $D_{F,1}A_{n,s}(W) - D_{F,1}Y_{n,s}(W)$ . Thus,  $D_{F,1}A_{n,s}(W)$  and  $D_{F,1}Y_{n,s}(W)$  are differentiable. As an immediate consequence we get (2.30) for  $X_n$  and  $Y_n$  instead of  $X$ .

For the second order Fréchet derivatives we repeat the whole part from (3.61) till the existence of first order Fréchet derivatives  $D_F A_{n,s}(W)$  and  $D_F Y_{n,s}(W)$  of  $A_{n,s}$  and  $Y_{n,s}$  in the space  $jH$ . This gives direction how to verify the existence of the corresponding second order Fréchet derivatives which we sketch below.

The difference is now that we deal with the mixed second order directional derivatives instead of the first order ones. For example, for the derivatives  $\frac{\partial^2}{\partial \kappa_i \partial \kappa_{i'}} A_{n,s}$  and  $\frac{\partial^2}{\partial \kappa_i \partial \kappa_{i'}} Y_{n,s}$  we modify relations (3.61) and (3.62) to

$$\begin{aligned}
Z_{i,i',s} & = \int_0^s \left\langle \left( \nabla_H^2(A - Y)_0(g'_n, \gamma_n) \right) (W^{n,u}), \right. \\
& \quad \left. \left( \left( H_i(\cdot + u), \frac{\partial}{\partial \kappa_i}(A_{n,u} - Y_{n,u}) \right), \left( H_{i'}(\cdot + u), \frac{\partial}{\partial \kappa_{i'}}(A_{n,u} - Y_{n,u}) \right) \right) \right\rangle_{H \otimes H \rightarrow F} du \\
& \quad + \int_0^s \left\langle \left( \nabla_H(A - Y)_0(g'_n, \gamma_n) \right) (W^{n,u}), (0, Z_{i,i',u}) \right\rangle_{H \rightarrow F} du \\
Z_{i,i',0} & = 0
\end{aligned}$$

with unique solution  $Z_{i,i',s} = \frac{\partial^2}{\partial \kappa_i \partial \kappa_{i'}}(A_{n,s} - Y_{n,s})$  and

$$\frac{\partial^2}{\partial \kappa_i \partial \kappa_{i'}} Y_{n,s} = \left\langle \left( \nabla_H^2 A_0(g_n, \gamma_n) \right) (W^{n,s}), \right.$$

$$\begin{aligned} & \left\langle \left( \left( H_i(\cdot + s), \frac{\partial}{\partial \kappa_i}(A_{n,s} - Y_{n,s}) \right), \left( H_{i'}(\cdot + s), \frac{\partial}{\partial \kappa_{i'}}(A_{n,s} - Y_{n,s}) \right) \right) \right\rangle_{H \otimes H \rightarrow F} \\ & + \int_0^s \left\langle \left( \nabla_H A_0(g_n, \gamma_n) \right) (W^{n,u}), \left( 0, \frac{\partial^2}{\partial \kappa_i \partial \kappa_{i'}}(A_{n,s} - Y_{n,s}) \right) \right\rangle_{H \rightarrow F}. \end{aligned}$$

For the well-finiteness of these relations we use condition (5) (ii) of Section 1. Moreover, for the convergence in  $H \otimes H$  of sums that correspond to (3.78) and (3.79), we pay particular attention to the crucial step for this, namely the counterpart to (3.63). In order to handle the second order partial derivatives which appear now we apply

$$\langle \nabla_H^2 \xi, (x, y) \rangle_{H \otimes H} = \lim_{\varepsilon, \delta \rightarrow 0} \frac{\xi(\cdot + \varepsilon x + \delta y) + \xi(\cdot - \varepsilon x - \delta y) - \xi(\cdot + \varepsilon x - \delta y) - \xi(\cdot - \varepsilon x + \delta y)}{4\varepsilon\delta}$$

if  $x \perp y$ ,  $\langle \nabla_H^2 \xi, (x, y) \rangle_{H \otimes H} = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} (\xi(\cdot + \varepsilon x) + \xi(\cdot - \varepsilon x) - 2\xi)$  if  $x = y$ , as well as linear combinations of these two formulas for general  $x, y$ , and we use condition (5) (i) of Section 1. Recalling Remark (2) of Section 2, we get the remaining part of condition (iii) of Proposition 2.7 for  $X_n$  and  $Y_n$  instead of  $X$ .  $\square$

**Parts (a) and (c) of Lemma 3.4 if parallel trajectories in  $\Omega$  do not necessarily generate identical jump times for  $X$ .** Let  $W \in \{\pi_m V : V \in \Omega\}$  with  $W_0 \notin G(W - W_0 \mathbb{I})$ , as in the above proofs of the parts (a) and (c) of Lemma 3.4 under the restriction of Remark (5) of this section. We concentrate on part (a). Part (c) can be treated in a similar way. Let  $\sigma(s, x) \equiv \sigma(s, x; W, x_0)$ ,  $s \in \mathbb{R}$ ,  $x \in F$ , be the time re-parametrization with respect to  $W - W_0 \mathbb{I}$  and  $W_0$  introduced in Remark (1) of Section 1.

*Step 1* We define a certain time substitution based on  $A_{n,\cdot}$ ,  $Y_{n,\cdot}$ , and  $\sigma$ . According to condition (2) (i) of Section 1 we may assume that  $s = 0$  is not a jump time. We set  $\eta(0) := 0$ ,  $\bar{A}_{n,s}(W) := A_0((W^{n,s})^{-s})$ , and  $\bar{Y}_{n,s}(W) := Y_0((W^{n,s})^{-s})$ ,  $s \in \mathbb{R}$ . We differentiate (3.27) with respect to  $s$  and multiply this coordinate wise with

$$d\eta(s) \equiv d\eta(s; n) := 2ds - d\sigma(s, W_0 + \bar{A}_{n,s} - \bar{Y}_{n,s}) \quad (3.86)$$

where  $\eta \equiv \eta(\cdot; n) : \mathbb{R} \rightarrow \mathbb{R}$  with  $\eta(0) = 0$  is supposed to be right continuous. Integrating the result we get

$$\begin{aligned} & \int_{r=0}^s \left( \dot{A}_{n,r}(W) - \dot{Y}_{n,r}(W) \right) d\eta(r) \\ &= \int_{r=0}^s \dot{C}_{n,0}(W^{n,r}) d\eta(r) \\ &= \int_{r=0}^s \int_F \int_F \left\langle \langle (A - Y)'_{-v}(W^{n,r} + y\mathbb{I}), g_n(v) \rangle_{F \rightarrow F}, \gamma_n(y) \right\rangle_{F \rightarrow F} dv dy d\eta(r) \\ & \quad + \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{r=0}^s \int_F \left\langle \langle \Delta(A - Y)_{\tau_k}(W^{n,r} + y\mathbb{I}), \right. \\ & \quad \left. g_n(\tau_k \circ u(W^{n,r} + y\mathbb{I})) \rangle_{F \rightarrow F} \gamma_n(y) \right\rangle_{F \rightarrow F} dy d\eta(r), \end{aligned}$$

$s \in \mathbb{R}$ , where we note that the integrands with respect to  $r$  of the above integrals are continuous by mollifying with  $\gamma_n$ .

Next, we compare the definitions of both mollifier functions  $g_n$  and  $\gamma_n$  in Subsection 1.2 preparing condition (5) therein. We give now a motivation for the order  $n^3$  in the definition of  $\gamma_n$ . We recall that for  $y < 1/n^3$  we have coordinate wise  $|g_n(v+y) - g_n(v)| \leq n^2 \max |g'_1| \cdot |y| \xrightarrow{n \rightarrow \infty} 0$ . This implies the existence of a sequence of real measurable functions  $\mathbb{R} \ni s \rightarrow c_n(s) \equiv c_n(s; W)$ ,  $n \in \mathbb{N}$ , with  $c_n \xrightarrow{n \rightarrow \infty} 0$  uniformly on every finite subinterval of  $\mathbb{R}$  such that

$$\begin{aligned} & \int_{r=0}^s \left( \dot{A}_{n,r}(W) - \dot{Y}_{n,r}(W) \right) d\eta(r) \\ &= \int_{r=0}^s \int \langle (A-Y)'_{-v}(W^{n,r}), g_n(v) \rangle_{F \rightarrow F} dv d\eta(r) \\ &+ \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{r=0}^s \langle \Delta(A-Y)_{\tau_k}(W^{n,r}), g_n(\tau_k \circ u(W^{n,r})) \rangle_{F \rightarrow F} d\eta(r) \\ &+ \int_{r=0}^s c_n(r) d\eta(r) \cdot \mathbf{e}, \quad s \in \mathbb{R}. \end{aligned} \tag{3.87}$$

Here, the integrand with respect to  $r$  of the second integral of the right-hand side is no longer necessarily continuous. However, by Definition 1.7 and the continuity of  $r \rightarrow W^{n,r}$  in the sense of (3.19) the functions  $r \rightarrow \Delta(A-Y)_{\tau_k}(W^{n,r})$  and  $r \rightarrow g_n(\tau_k \circ u(W^{n,r}))$  are piecewise continuous. By the asymptotic behavior of the mollifier function  $\gamma_n$  as  $n \rightarrow \infty$  the increase of the second integral of the right-hand side at jump times is now appropriately defined as follows. Form coordinate wise the product of the increase of  $\eta$  times the average of the left and the right limit of the integrand at jump points. Note that these limits exist by Definition 1.7 (j).

For the moment, fix  $n \in \mathbb{N}$  and, without loss of generality,  $s > 0$ . By (3.86) and property (i) of Remark (1) in Section 1, the function  $\eta(r)$  is piecewise continuously differentiable in  $r \in [0, s)$ . Thus the function  $\eta$  has on  $[0, s)$  countably many disjoint real intervals  $(a_l^+, b_l^+) \subseteq [0, s)$  of strict continuous increase resp. strict continuous decrease  $(a_l^-, b_l^-) \subseteq [0, s)$ ,  $l \in \mathbb{N}$ . On these intervals  $\eta(r)$  is continuously differentiable.

In addition  $\eta$  has on  $[0, s)$  countably many positive jumps, say at  $J_i^+ \in [0, s)$  and countably many negative jumps, say at  $J_i^- \in [0, s)$ . By property (i) of Remark (1) in Section 1,  $\eta$  has left and right limits at these jump times.

For each of the intervals of strict continuous increase or decrease, the function  $\eta$  has a well-defined *local* inverse  $\eta_{l,+}^{-1}$  resp.  $\eta_{l,-}^{-1}$ . In other words, there exist countably many not necessarily disjoint intervals  $(\alpha_l^+, \beta_l^+)$  resp.  $(\alpha_l^-, \beta_l^-)$ ,  $l \in \mathbb{N}$ , such that  $\eta_{l,+}^{-1}$  maps  $(\alpha_l^+, \beta_l^+)$  onto  $(a_l^+, b_l^+)$  resp.  $\eta_{l,-}^{-1}$  maps  $(\alpha_l^-, \beta_l^-)$  onto  $(a_l^-, b_l^-)$ . Furthermore, there are real bounded open intervals  $I_i^+$  resp.  $I_i^-$  such that  $\eta(J_i^+)- = \inf I_i^+$  and  $\eta(J_i^+)+ = \sup I_i^+$  resp.  $\eta(J_i^-)- = \sup I_i^-$  and  $\eta(J_i^-)+ = \inf I_i^-$ . We introduce corresponding *local* inverse maps  $\eta_{J_i^+}^{-1}$  and  $\eta_{J_i^-}^{-1}$  by  $J_i^+ = \eta_{J_i^+}^{-1}(r)$ ,  $r \in I_i^+$ , and  $J_i^- = \eta_{J_i^-}^{-1}(r)$ ,  $r \in I_i^-$ . To simplify the notation, we summarize the intervals and points

$$\mathcal{I} := \{ \text{all } (a_l^+, b_l^+), (a_l^-, b_l^-) \text{ and all } J_i^+, J_i^- \}$$

and denote for each  $I \in \mathcal{I}$  the above local inverse map by  $\bar{\eta}^{-1} \equiv \bar{\eta}_I^{-1}$ . If  $I \in \mathcal{I}$  is an interval, the inverse  $\bar{\eta}$  to  $\bar{\eta}_I^{-1}$  is the restriction of  $\eta$  to  $I$ . If  $I \in \mathcal{I}$  is a point, let the *formal* inverse  $\bar{\eta}$  to  $\bar{\eta}_I^{-1}$  be defined by  $\bar{\eta}(J_i^+) := \sup I_i^+ = \eta(J_i^+)+$  and  $\bar{\eta}(J_i^-) := \inf I_i^- = \eta(J_i^-)+$ .

*Step 2* We discover some of properties of the time substitution  $\eta$ . Let  $r \in [0, s)$  and  $k \in \mathbb{Z} \setminus \{0\}$ . We take into consideration that by definition  $\eta(r) = 2r - \sigma(r, W_0 + \bar{A}_{n,r} - \bar{Y}_{n,r})$  and

$$\begin{aligned} \tau_k \circ u(W^{n,r}) + r &= \tau_{k'} \circ u(W + (\bar{A}_{n,r} - \bar{Y}_{n,r})\mathbb{1}) \\ &= \sigma(\tau_{k'} \circ u(W), W_0 + \bar{A}_{n,r} - \bar{Y}_{n,r}), \quad r \in [0, s), \end{aligned} \quad (3.88)$$

for some  $k' \equiv k'(k; W, r) \in \mathbb{Z} \setminus \{0\}$  which we specify below, cf. also Remark (1) (v) of Section 1. According to Definition 1.7 (jjj), for all  $k \in \mathbb{Z} \setminus \{0\}$  such that  $\tau_{k'} \circ u(W) \in [0, s)$  it holds that

$$\begin{aligned} \tau_k \circ u(W^{n,r}) + r &= \sigma(\tau_{k'} \circ u(W), W_0 + \bar{A}_{n,r} - \bar{Y}_{n,r}) \\ &= \sigma(r, W_0 + \bar{A}_{n,r} - \bar{Y}_{n,r}) \\ &\quad + \sigma(\tau_{k'} \circ u(W), W_0 + \bar{A}_{n,r} - \bar{Y}_{n,r}) - \sigma(r, W_0 + \bar{A}_{n,r} - \bar{Y}_{n,r}) \\ &= \sigma(r, W_0 + \bar{A}_{n,r} - \bar{Y}_{n,r}) + \tau_{k'} \circ u(W) - r \\ &= 2r - \eta(r) + \tau_{k'} \circ u(W) - r = r - \eta(r) + \tau_{k'} \circ u(W) \quad \text{if} \quad |\tau_k \circ u(W^{n,r})| \leq \frac{1}{n} \end{aligned}$$

for sufficiently large  $n \in \mathbb{N}$ . Summarizing the last paragraph, we get

$$\tau_k \circ u(W^{n,r}) = -\eta(r) + \tau_{k'} \circ u(W) \quad \text{if} \quad |\tau_k \circ u(W^{n,r})| \leq \frac{1}{n} \quad (3.89)$$

for sufficiently large  $n \in \mathbb{N}$ . We mention that in this case  $\tau_k \circ u(W^r) = -r + \tau_{k'} \circ u(W)$  which means that  $k$  and  $k'$  are related as in Step 2 of the proof of part (a) of Lemma 3.4 under the restriction of Remark (5) of the present section.

It follows from (v) of Remark (1) of Section 1 and (jjj) of Definition 1.7 that for  $k'$  as above

$$\begin{aligned} 2(\tau_{k'} \circ u(W + (x - W_0)\mathbb{1}) + \delta) - \sigma(\tau_{k'} \circ u(W + (x - W_0)\mathbb{1}) + \delta, x) \\ = \tau_{k'} \circ u(W) + \delta, \quad x \in D^n, \end{aligned} \quad (3.90)$$

where we keep in mind that we take here  $W - W_0\mathbb{1}$  for  $W$  in Definition 1.7 and  $W_0$  here for  $x_0$  there. We pick up the situation in (3.89), i. e., we assume that  $r$  is chosen such that  $|\tau_k \circ u(W^{n,r})| \leq \frac{1}{n}$  for sufficiently large  $n \in \mathbb{N}$ . We set  $\delta := -\tau_k \circ u(W^{n,r})$ . It follows from (3.88) that  $r = \tau_{k'} \circ u(W + (\bar{A}_{n,r} - \bar{Y}_{n,r})\mathbb{1}) + \delta$ . With  $x := W_0 + \bar{A}_{n,r} - \bar{Y}_{n,r}$  we obtain from (3.90)

$$\begin{aligned} \eta(\tau_{k'} \circ u(W + (\bar{A}_{n,r} - \bar{Y}_{n,r})\mathbb{1}) + \delta) &= \eta(r) \\ &= 2r - \sigma(r, W_0 + \bar{A}_{n,r} - \bar{Y}_{n,r}) \\ &= \tau_{k'} \circ u(W) + \delta \\ &= \eta(\tau_{k'} \circ u(W + (\bar{A}_{n,\tau_{k'}} - \bar{Y}_{n,\tau_{k'}})\mathbb{1})) + \delta \end{aligned} \quad (3.91)$$

where  $\tau_{k'}$  without any further argument stands until the end of the present Step 2 for  $\tau_{k'} \circ u(W + (\bar{A}_{n,\tau_{k'}} - \bar{Y}_{n,\tau_{k'}})\mathbb{1})$ , i. e., the last equality sign is (3.89) for  $r \equiv r_0 =: \tau_{k'} \circ$

$u(W + (\bar{A}_{n,\tau_{k'}} - \bar{Y}_{n,\tau_{k'}}) \mathbb{1})$ . We note that for this particular  $r_0$  we have  $\tau_k \circ u(W^{n,r_0}) = 0$ . For  $I \in \mathcal{I}$  being an interval and  $\tau_{k'} \circ u(W + (\bar{A}_{n,\tau_{k'}} - \bar{Y}_{n,\tau_{k'}}) \mathbb{1}) \in I$  we obtain

$$\bar{\eta}_I^{-1}(\tau_{k'} \circ u(W)) = \tau_{k'} \circ u(W + (\bar{A}_{n,\tau_{k'}} - \bar{Y}_{n,\tau_{k'}}) \mathbb{1})$$

and

$$\bar{\eta}_I^{-1}(\tau_{k'} \circ u(W) + \delta) = r = \tau_{k'} \circ u(W + (\bar{A}_{n,r} - \bar{Y}_{n,r}) \mathbb{1}) + \delta.$$

Recalling  $\delta \equiv \delta_k(r) = -\tau_k \circ u(W^{n,r})$  this leads to

$$\frac{d}{d\delta} \bar{\eta}_I^{-1}(\tau_{k'} \circ u(W) + \delta) = \frac{dr}{d\delta} = 1 + \frac{d\tau_{k'} \circ u(W + (\bar{A}_{n,r} - \bar{Y}_{n,r}) \mathbb{1})}{dr} \Big/ \frac{d\delta_k(r)}{dr} \quad (3.92)$$

for sufficiently large  $n \in \mathbb{N}$  which means that we assume  $|\tau_k \circ u(W^{n,r})| \leq \frac{1}{n}$ .

*Step 3* Setting  $p = \tau_{k'} \circ u(W) + \delta$  and looking at (3.92), the goal of this step is an estimate on  $|\bar{d}\bar{\eta}_I^{-1}(p)/dp - 1|$ . For the subsequent calculations we recall condition (3) (i) and Remark (3) of Section 1. Furthermore, we keep Lemma 3.4 (vi) in mind. In particular it holds that

$$\begin{aligned} \bar{A}_{n,r}(W) - \bar{Y}_{n,r}(W) &= A_{n,r}(W) - Y_{n,r}(W) + (A - Y)_{-r}(W^{n,r}) \\ &= \left( A_{n,r}(W) - Y_{n,r}(W) - (A - Y)_r(W) \right) + \left( (A - Y)_{-r}(W^{n,r}) - (A - Y)_{-r}(W^r) \right), \end{aligned} \quad (3.93)$$

As in (3.28) let  $\psi(r) = A_{n,r}(W) - Y_{n,r}(W) - (A - Y)_r(W)$ . From the convention that 0 is not a jump time for  $X$  it follows that  $-r$  is not a jump time for  $u(W^r)$ . For sufficiently large  $n \in \mathbb{N}$  we may because of  $|\tau_k \circ u(W^{n,r})| \leq \frac{1}{n}$  also assume that  $-r$  is not a jump time for  $u(W^{n,r})$ . Using (3.93) we obtain for sufficiently large  $n \in \mathbb{N}$  and  $r \in [0, s)$  not being a jump time for  $u(W) = X$  such that  $|\tau_k \circ u(W^{n,r})| \leq \frac{1}{n}$

$$\begin{aligned} \frac{d}{dr} (\bar{A}_{n,r} - \bar{Y}_{n,r})(W) &= \frac{d}{dr} \left( (A_{n,r} - Y_{n,r})(W) - (A - Y)_{-r}(W^{n,r}) \right) \\ &= \left( (\dot{A}_{n,r} - \dot{Y}_{n,r})(W) - (A - Y)'_r(W) \right) + \left( (A - Y)'_{-r}(W^{n,r}) - (A - Y)'_{-r}(W^r) \right) \\ &\quad - \left( \lim_{h \rightarrow 0} \frac{1}{h} ((A - Y)_{-r}(W^{n,r+h}) - (A - Y)_{-r}(W^{n,r})) \right. \\ &\quad \left. - \lim_{h \rightarrow 0} \frac{1}{h} ((A - Y)_{-r}(W^{r+h}) - (A - Y)_{-r}(W^r)) \right). \end{aligned}$$

We treat the three differences on the right-hand side, separated by parentheses, separately. For the first one we form the difference with the left-hand side and recall how we have applied (3.27) in Step 1 of this proof. Furthermore, we take into consideration that  $(A - Y)'_r(W) = (A - Y)'_0(W^r)$ . For the second one, we use the idea of (3.31). In order to treat the third difference we recall the way we have used Proposition 2.7 in Lemma 3.2 (b) and (c). Again with the idea of estimating used in (3.31) we get

$$\left| \frac{d}{dr} (\bar{A}_{n,r} - \bar{Y}_{n,r})(W) - \sum_{k \in \mathbb{Z} \setminus \{0\}} \langle \Delta(A - Y)_{\tau_k}(W^{n,r}), g_n(\tau_k \circ u(W^{n,r})) \rangle_{F \rightarrow F} \right|$$

$$\leq \left| (A - Y)'_0(W^{n,r}) - (A - Y)'_0(W^r) \right| + |c_n(r)| + \alpha'(S; W) \cdot |\psi(r)| + \delta'(S; W) \cdot |\psi(r)| \quad (3.94)$$

for some positive constant  $\delta'(S; W) < \infty$  where  $S = [0, s)$ . In (3.94) and in the next relation the absolute values are taken coordinate wise. Furthermore by (3.93),

$$\begin{aligned} & \left| (\nabla_{W_0} \tau_{k'} \circ u) (W + (\bar{A}_{n,r}(W) - \bar{Y}_{n,r}(W)) \mathbb{1}) - (\nabla_{W_0} \tau_{k'} \circ u) (W) \right| \\ & \leq \kappa'(S; W) \cdot |\psi(r)| \end{aligned} \quad (3.95)$$

for some positive constant  $\kappa'(S; W) < \infty$  by (j) of Definition 1.7 and condition (3) (ii) of Section 1. For sufficiently large  $n \in \mathbb{N}$  and  $r \in [0, s)$  not being a jump time for  $u(W) = X$  such that  $|\tau_k \circ u(W^{n,r})| \leq \frac{1}{n}$  the subsequent estimate uses (3.94) and (3.95) and

$$\langle \tau_{k'} \circ u (W + (\bar{A}_{n,r}(W) - \bar{Y}_{n,r}(W)) \mathbb{1}) , \Delta(A - Y)_{\tau_k} (W^{n,r}) \rangle_F = 0 ,$$

recall  $(W + (\bar{A}_{n,r}(W) - \bar{Y}_{n,r}(W)) \mathbb{1})^r = W^{n,r}$ ,  $\tau_{k'} \circ u ((W + (\bar{A}_{n,r}(W) - \bar{Y}_{n,r}(W)) \mathbb{1})^r) = \tau_k \circ u (W^{n,r})$ , and (jv) of Definition 1.7. We get the estimate

$$\begin{aligned} & \left| \frac{d}{dr} \tau_{k'} \circ u (W + (\bar{A}_{n,r}(W) - \bar{Y}_{n,r}(W)) \mathbb{1}) \right| \\ & = \left| \left\langle (\nabla_{W_0} \tau_{k'} \circ u) (W + (\bar{A}_{n,r}(W) - \bar{Y}_{n,r}(W)) \mathbb{1}) , \right. \right. \\ & \quad \left. \frac{d}{dr} (\bar{A}_{n,r} - \bar{Y}_{n,r}) (W) - \sum_{k \in \mathbb{Z} \setminus \{0\}} \langle \Delta(A - Y)_{\tau_k} (W^{n,r}) , g_n (\tau_k \circ u(W^{n,r})) \rangle_{F \rightarrow F} \right\rangle_F \\ & \quad + \sum_{k \in \mathbb{Z} \setminus \{0\}} \left\langle \tau_{k'} \circ u (W + (\bar{A}_{n,r}(W) - \bar{Y}_{n,r}(W)) \mathbb{1}) , \right. \\ & \quad \left. \langle \Delta(A - Y)_{\tau_k} (W^{n,r}) , g_n (\tau_k \circ u(W^{n,r})) \rangle_{F \rightarrow F} \right\rangle_F \left. \right| \\ & \leq \langle |(\nabla_{W_0} \tau_{k'} \circ u) (W)| , |c_n(r)| + 2\alpha'(S; W) \cdot |\psi(r)| + \delta'(S; W) \cdot |\psi(r)| \rangle_F \\ & \quad + \langle \kappa'(S; W) \cdot |\psi(r)| , |c_n(r)| + 2\alpha'(S; W) \cdot |\psi(r)| + \delta'(S; W) \cdot |\psi(r)| \rangle_F . \end{aligned}$$

For this we recall also that for the whole proof we hold  $W \in \{\pi_m V : V \in \Omega\}$  with  $W_0 \notin G(W - W_0 \mathbb{1})$  fixed. Summing up, we have

$$\left| \frac{d}{dr} \tau_{k'} \circ u (W + (\bar{A}_{n,r}(W) - \bar{Y}_{n,r}(W)) \mathbb{1}) \right| \leq C_1 \cdot |\psi(r)| + C_2 \cdot |\psi(r)|^2 + d_n(r) \quad (3.96)$$

for some positive constants  $C_1 \equiv C_1(S; W) < \infty$ ,  $C_2 \equiv C_2(S; W) < \infty$  and  $d_n(r) \xrightarrow{n \rightarrow \infty} 0$  boundedly on  $r \in [0, s) \equiv S$ . As above we mention that (3.96) holds for sufficiently large

$n \in \mathbb{N}$  and for  $r \in [0, s)$  such that  $|\tau_k \circ u(W^{n,r})| \leq \frac{1}{n}$ . The restriction not being a jump time for  $u(W) = X$  can now be neglected by looking at  $r = \tau_{k'} \circ u(W + (\bar{A}_{n,r} - \bar{Y}_{n,r}) \mathbb{1}) + \delta$  again and recalling that  $\delta \equiv \delta_k(r) = -\tau_k \circ u(W^{n,r})$ .

In order to prepare the conclusions of this step we recall (3.89) which in particular says that  $p = \delta + \tau_{k'} \circ u(W) = \eta(r)$ . We note that  $\psi(r) \equiv \psi(r; n, W)$  is bounded in  $n \in \mathbb{N}$  on  $r \in [0, s) = S$ . This is a consequence of the last sentence in Lemma 3.4 (vi).

We observe that, without loss of generality, we may assume  $|\liminf_r d\delta_k(r)/dr| > 0$  for  $r$  such that  $|\tau_k \circ u(W^{n,r})| \leq \frac{1}{n}$  and  $n \in \mathbb{N}$  sufficiently large. This assumption is justified by the fact that supposing  $d\delta_k(r)/dr = -d\tau_k \circ u(W^{n,r})/dr \rightarrow 0$  on some subsequence  $n_k \rightarrow \infty$  and some  $r \equiv r_{n_k}$  with the above hypotheses on  $r$ , we would by (3.27) obtain  $|\psi(r)| \rightarrow \infty$  on this subsequence. However we have already given an argument that this is impossible.

For some  $C_3 \equiv C_3(W; S)$  independent of  $n \in \mathbb{N}$  we obtain ow from (3.92) and (3.96)

$$\left| \frac{d\bar{\eta}_I^{-1}(p)}{dp} - 1 \right| \leq C_3 \cdot |\psi(\bar{\eta}_I^{-1}(p))| + d_n(\bar{\eta}_I^{-1}(p)) \quad (3.97)$$

for all  $p \in \{\eta(r') : r' \in I\} \cap [\tau_{k'} \circ u(W) - \frac{1}{n}, \tau_{k'} \circ u(W) + \frac{1}{n}]$  and all  $k' \in \mathbb{Z} \setminus \{0\}$  such that  $\tau_{k'} \circ u(W + (\bar{A}_{n,\tau_{k'}} - \bar{Y}_{n,\tau_{k'}}) \mathbb{1}) \in I$  where  $I \in \mathcal{I}$  is an interval.

Moreover, there exist  $b_n(I) > 0$  for all intervals  $I \in \mathcal{I}$  such that the sum of the  $b_n(I)$  over all such  $I$  is finite and  $b_n(I) \xrightarrow{n \rightarrow \infty} 0$  boundedly and

$$\int \left| \frac{d\bar{\eta}_I^{-1}(p)}{dp} - 1 \right| dp \leq b_n(I). \quad (3.98)$$

for sufficiently large  $n \in \mathbb{N}$ . Here the integral ranges over all  $p \in \{\eta(r') : r' \in I\} \setminus [\tau_{k'} \circ u(W) - \frac{1}{n}, \tau_{k'} \circ u(W) + \frac{1}{n}]$ . This is a consequence of the following observation. For time intervals between two jumps of the form  $\tau_k \circ u(W^{n,r})$ , (3.27) can be treated in a similar manner, no matter whether or not we require the restriction of Remark (5) of this section.

*Step 4* We introduce a decomposition similar to (3.27)-(3.30). Let  $I \in \mathcal{I}$ ,  $q \in I$ , and if  $I$  is just a point, set  $\inf I := I -$ . For the sake of formality, we write also  $\bar{\eta}(\inf I)$  for  $\eta(\inf I)$ . Using (3.87) and (3.89) we obtain for  $k' \equiv k'(k; W, \bar{\eta}^{-1}(r))$  and sufficiently large  $n \in \mathbb{N}$

$$\begin{aligned} \int_{\inf I}^q \left( \dot{A}_{n,r}(W) - \dot{Y}_{n,r}(W) \right) d\eta(r) &= \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} \left( \dot{A}_{n,\bar{\eta}^{-1}(r)}(W) - \dot{Y}_{n,\bar{\eta}^{-1}(r)}(W) \right) dr \\ &= \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} \int \left\langle (A - Y)'_{-v}(W^{n,\bar{\eta}^{-1}(r)}), g_n(v) \right\rangle_{F \rightarrow F} dv dr \\ &\quad + \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} \left\langle \Delta(A - Y)_{\tau_k}(W^{n,\bar{\eta}^{-1}(r)}), g_n(\tau_{k'} \circ u(W) - r) \right\rangle_{F \rightarrow F} dr \\ &\quad + \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} c_n(\bar{\eta}^{-1}(r)) dr \cdot \mathbf{e} \end{aligned} \quad (3.99)$$

where we note that the variable substitution is compatible with the above mode of integration with respect to  $d\eta$  at jump times. We set

$$\begin{aligned} \hat{r}(r) \equiv \hat{r}(r; n, W) &:= \int \left\langle (A - Y)'_{-v}(W^{\bar{\eta}^{-1}(r)}), g_n(v) \right\rangle_{F \rightarrow F} dv \\ &\quad + \sum_{k \in \mathbb{Z} \setminus \{0\}} \left\langle \Delta(A - Y)_{\tau_k}(W^{\bar{\eta}^{-1}(r)}), g_n(\tau_{k'} \circ u(W) - r) \right\rangle_{F \rightarrow F} + c_n(\bar{\eta}^{-1}(r)) \cdot \mathbf{e}. \end{aligned}$$

Differentiating (3.99) and re-integrating then with respect to  $\bar{\eta}^{-1}$  we get for  $I \in \mathcal{I}$  and  $q \in I$

$$\begin{aligned}
& (A - Y)_{n,q}(W) - (A - Y)_{n,\inf I}(W) \\
& \quad - ((A - Y)_q(W) - (A - Y)_{\inf I}(W)) \\
& = \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} \left( \dot{A}_{n,\bar{\eta}^{-1}(r)}(W) - \dot{Y}_{n,\bar{\eta}^{-1}(r)}(W) \right) d\bar{\eta}^{-1}(r) \\
& \quad - ((A - Y)_q(W) - (A - Y)_{\inf I}(W)) \\
& = \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} \int \left\langle (A - Y)'_{-v} \left( W^{n,\bar{\eta}^{-1}(r)} \right) - (A - Y)'_{-v} \left( W^{\bar{\eta}^{-1}(r)} \right), \right. \\
& \quad \left. g_n(v) \right\rangle_{F \rightarrow F} dv d\bar{\eta}^{-1}(r) \\
& \quad + \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} \left\langle \Delta(A - Y)_{\tau_k} \left( W^{n,\bar{\eta}^{-1}(r)} \right) - \Delta(A - Y)_{\tau_k} \left( W^{\bar{\eta}^{-1}(r)} \right), \right. \\
& \quad \left. g_n(\tau_{k'} - r) \right\rangle_{F \rightarrow F} d\bar{\eta}^{-1}(r) \\
& \quad + \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} \hat{r}(r) d\bar{\eta}^{-1}(r) - ((A - Y)_q(W) - (A - Y)_{\inf I}(W)) \tag{3.100}
\end{aligned}$$

where, in case that  $I$  is just a point, we read the integral  $\int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} (\cdot)(\bar{\eta}^{-1}(r)) d\bar{\eta}^{-1}(r)$  as  $\int_{\inf I}^q (\cdot)(r) dr = 0$ . Now we recall that the time re-parametrization  $\sigma(s, x)$ ,  $s \in \mathbb{R}$ ,  $x \in D^n$ , with respect to  $W - W_0 \mathbb{1}$  and  $W_0$  of Remark (1) in Section 1 is on every finite subinterval of  $\mathbb{R}$  just in the time points  $\tau_{k'} \circ u(W)$  (and all  $x \in D^n$ ) uniquely defined, cf. (v) of Remark (1) in Section 1.

*Step 5* We construct estimates on the items of (3.100). Using (3.86) and (3.91) we observe the following. Except for neighborhoods of the finitely many jumps times  $\tau_{k'} \circ u(W + (\bar{A}_{n,\tau_{k'}} - \bar{Y}_{n,\tau_{k'}}) \mathbb{1}) \in [0, s)$  such that they correspond via  $\eta$  to  $\varepsilon$ -neighborhoods of  $\tau_{k'} \circ u(W)$  in the sense of (3.91), the functions  $\eta \equiv \eta(\cdot; n)$  can be slightly altered on  $[0, s)$ . For  $\varepsilon > 0$  let  $\varepsilon(S)$  denote the union of these neighborhoods of  $\tau_{k'} \circ u(W + (\bar{A}_{n,\tau_{k'}} - \bar{Y}_{n,\tau_{k'}}) \mathbb{1})$  intersected with  $S = [0, s)$ .

In particular, we may assume that for no  $n \in \mathbb{N}$  there is some interval of constancy on  $[0, s) \setminus \varepsilon([0, s))$ . Without loss of generality, we may also assume that for all  $I \in \mathcal{I}$  being intervals the restriction of  $\bar{\eta}_I^{-1}$  to  $\{\bar{\eta}_I(r) : r \in I \setminus \varepsilon([0, s))\}$  is continuously differentiable.

We set

$$\hat{\psi}(r) \equiv \hat{\psi}(r; n, W) := A_{n,\bar{\eta}^{-1}(r)}(W) - Y_{n,\bar{\eta}^{-1}(r)}(W) - (A - Y)_{\bar{\eta}^{-1}(r)}(W).$$

Recalling how we proceeded in Steps 2 and 3 of the proof of part (a) of Lemma 3.4 under the restriction of Remark (5) of this section, we arrive with (3.100) at

$$(A - Y)_{n,q}(W) - (A - Y)_{n,\inf I}(W) - ((A - Y)_q(W) - (A - Y)_{\inf I}(W))$$

$$\begin{aligned}
&\leq \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} \left( \alpha'(S; W) \cdot |\hat{\psi}(r)| + \gamma'(S; W) \sum_{k \in \mathbb{Z} \setminus \{0\}} \left\langle |\hat{\psi}(r)|, g_n(\tau_{k'} - r) \right\rangle_{F \rightarrow F} \right) d\bar{\eta}^{-1}(r) \\
&+ \left| \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} \int \left\langle (A - Y)'_{-v} \left( W^{\bar{\eta}^{-1}(r)} \right), g_n(v) \right\rangle_{F \rightarrow F} dv d\bar{\eta}^{-1}(r) \right. \\
&+ \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} \left\langle \Delta(A - Y)_{\tau_k} \left( W^{\bar{\eta}^{-1}(r)} \right), g_n(\tau_{k'} - r) \right\rangle_{F \rightarrow F} d\bar{\eta}^{-1}(r) \\
&\left. - ((A - Y)_q(W) - (A - Y)_{\inf I}(W)) \right| + \int_{\inf I}^q |c_n(r)| dr \cdot \mathbf{e} \\
&=: R_1 + |T_1| + R_2
\end{aligned} \tag{3.101}$$

where  $R_1 \equiv R_1(n; I)$  is the term in the first line after the “ $\leq$ ” sign,  $T_1 \equiv T_1(n; I)$  is the term inside the coordinate wise absolute value signs that follows  $R_1$ .  $R_2 \equiv R_2(n; I)$  is the last integral on the right-hand side. Because of (3.89) we have for all  $I \in \mathcal{I}$  and sufficiently large  $n \in \mathbb{N}$

$$\tau_k \circ u(W^{n, \bar{\eta}^{-1}(r)}) = -r + \tau_{k'} \circ u(W) = \tau_k \circ u(W^r)$$

or, equivalently,

$$\tau_k \circ u(W^{n, \bar{\eta}^{-1}(r)}) - \tau_k \circ u(W^{\bar{\eta}^{-1}(r)}) = \tau_k \circ u(W^r) - \tau_k \circ u(W^{\bar{\eta}^{-1}(r)})$$

if  $\left| \tau_k \circ u(W^{n, \bar{\eta}^{-1}(r)}) \right| \leq \frac{1}{n}$  which implies  $|-r + \tau_{k'} \circ u(W)| \leq \frac{1}{n}$ . This yields

$$\left| \Delta(A - Y)_{\tau_k} \left( W^{\bar{\eta}^{-1}(r)} \right) - \Delta(A - Y)_{\tau_k} (W^r) \right| \leq \gamma'(S; W) \cdot |\hat{\psi}(r)|$$

and therefore

$$\begin{aligned}
T_1 &\leq \left| \int_{\inf I}^q \int \left\langle (A - Y)'_{-v} (W^r), g_n(v) \right\rangle_{F \rightarrow F} dv dr \right. \\
&+ \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} \left\langle \Delta(A - Y)_{\tau_k} (W^r), g_n(\tau_{k'} - r) \right\rangle_{F \rightarrow F} d\bar{\eta}^{-1}(r) \\
&\left. - ((A - Y)_q(W) - (A - Y)_{\inf I}(W)) \right| \\
&+ \gamma'(S; W) \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} \left\langle |\hat{\psi}(r)|, g_n(\tau_{k'} - r) \right\rangle_{F \rightarrow F} d\bar{\eta}^{-1}(r) \\
&=: |T_2| + R_3
\end{aligned} \tag{3.102}$$

where  $T_2 \equiv T_2(n; I)$  is the term inside the coordinate wise absolute value signs after the “ $\leq$ ” sign.  $R_3 \equiv R_3(n; I)$  is the last line on the right-hand side. Furthermore, if  $I$  is an interval,

we have

$$\begin{aligned}
& \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} \int_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle (A - Y)'_{r-v}(W), g_n(v) \right\rangle_{F \rightarrow F} dv dr \\
& \quad + \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} \sum_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle \Delta(A - Y)_{r-v}(W), g_n(v) \right\rangle_{F \rightarrow F} dr \\
& = \int_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle (A - Y)_{\bar{\eta}(q)-v}(W) - (A - Y)_{\bar{\eta}(\inf I)-v}(W), g_n(v) \right\rangle_{F \rightarrow F} dv
\end{aligned}$$

and consequently

$$\begin{aligned}
T_2 \leq & \left| \left( \int_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle (A - Y)_{\bar{\eta}(q)-v}(W) - (A - Y)_{\bar{\eta}(\inf I)-v}(W), g_n(v) \right\rangle_{F \rightarrow F} dv \right. \right. \\
& - \left. \left( (A - Y)_{\bar{\eta}(q)}(W) - (A - Y)_{\bar{\eta}(\inf I)}(W) \right) \right. \\
& + \left. \left( \int_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle \int_{\inf I}^q (A - Y)'_{r-v}(W) dr - \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} (A - Y)'_{r-v}(W) dr, g_n(v) \right\rangle_{F \rightarrow F} dv \right. \right. \\
& + \left. \int_{\inf I}^q \sum_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle \Delta(A - Y)_{r-v}, g_n(v) \right\rangle_{F \rightarrow F} dr \right. \\
& \quad \left. - \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} \sum_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle \Delta(A - Y)_{r-v} dr, g_n(v) \right\rangle_{F \rightarrow F} \right. \\
& - \left. \left( (A - Y)_q(W) - (A - Y)_{\inf I}(W) \right) + \left( (A - Y)_{\bar{\eta}(q)}(W) - (A - Y)_{\bar{\eta}(\inf I)}(W) \right) \right. \\
& - \left. \left( \int_{\inf I}^q \sum_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle \Delta(A - Y)_{r-v}, g_n(v) \right\rangle_{F \rightarrow F} dr \right. \right. \\
& \quad \left. \left. - \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} \sum_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle \Delta(A - Y)_{r-v} dr, g_n(v) \right\rangle_{F \rightarrow F} \right) \right| + R_4 \\
& = |T_{31} + T_{32} - Q| + R_4 \tag{3.103}
\end{aligned}$$

where  $T_{31} \equiv T_{31}(n; I)$ ,  $T_{32} \equiv T_{32}(n; I)$ , and  $Q \equiv Q(n; I)$  are the terms inside the coordinate wise absolute value signs separated by parentheses and

$$R_4 \equiv R_4(n; I) := \left| \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} \left\langle \Delta(A - Y)_{\tau_k}(W^r), g_n(\tau_{k'} - r) \right\rangle_{F \rightarrow F} d(\bar{\eta}^{-1}(r) - r) \right|. \tag{3.104}$$

*Step 6* In this step we look at (3.101)-(3.104) simultaneously for all  $I \in \mathcal{I}$ . By definition, the most right  $I \in \mathcal{I}$  is an interval. For that  $I$  we choose  $q := s$ . For all other intervals

$I \in \mathcal{I}$ , let  $q := \sup I$ . If  $I$  is just a point then let  $q := I$ . In this case we remind of the convention  $\int_{\bar{\eta}(\inf I)}^{\bar{\eta}(q)} (\cdot)(\bar{\eta}^{-1}(r)) d\bar{\eta}^{-1}(r) = \int_{\inf I}^q (\cdot)(r) dr = 0$ . Moreover, if  $I = q$  is just a point, we have  $R_1 = R_2 = R_3 = 0$  and  $T_1 = T_2 = -\Delta(A - Y)_q(W)$ . If  $I$  is just a point, we set  $T_{31} := -((A - Y)_{\bar{\eta}(q)}(W) - (A - Y)_{\bar{\eta}(\inf I)}(W))$ ,  $T_{32} := -((A - Y)_q(W) - (A - Y)_{\inf I}(W) + ((A - Y)_{\bar{\eta}(q)}(W) - (A - Y)_{\bar{\eta}(\inf I)}(W)))$ , and consequently  $R_4 := 0$ .

Recalling (3.97) we get with some positive constant  $C_4 \equiv C_4(S; W) < \infty$  not depending on  $n \in \mathbb{N}$  the estimate

$$\begin{aligned} & \sum_{I \in \mathcal{I} \text{ is interval}} R_4(n; I) \\ & \leq C_4 \cdot \sum_{I \in \mathcal{I} \text{ is interval}} \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{\bar{\eta}(\inf I)}^{\bar{\eta}(\sup I)} \left\langle g_n(\tau_{k'} - p), \left( |\hat{\psi}(p)| + d_n(\bar{\eta}^{-1}(p)) \cdot \mathbf{e} \right) \right\rangle_{F \rightarrow F} dp \end{aligned} \quad (3.105)$$

where for the usage of (3.97) we have taken into consideration that  $\hat{\psi}(p) = \psi(\bar{\eta}^{-1}(p))$ .

We complete the proof by referring to the similarities in (3.33)-(3.35). First we take a look at  $\sum_{I \in \mathcal{I}} T_{31}(n; I)$  and  $\sum_{I \in \mathcal{I}} T_{32}(n; I)$ , where  $T_{31}$  and  $T_{32}$  are defined in (3.103). Here we point out similarities to (3.34) if  $I$  is an interval. For the choice of  $q = I$  in case that  $I$  is just a point recall the first paragraph of the present step. As in (3.34) it turns out that

$$\hat{\rho}_1(s; n, W) := \sum_{I \in \mathcal{I}} T_{31}(n; I) \xrightarrow{n \rightarrow \infty} 0. \quad (3.106)$$

In addition, assuming without loss of generality  $\eta(s) \leq s$ ,

$$\begin{aligned} \hat{\rho}_2(s; n, W) &:= \sum_{I \in \mathcal{I}} T_{32}(n; I) = \int_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle \int_{\eta(s)}^s (A - Y)'_{r-v}(W) dr, g_n(v) \right\rangle_{F \rightarrow F} dv \\ &+ \int_{\eta(s)}^s \sum_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle \Delta(A - Y)_{r-v}(W), g_n(v) \right\rangle_{F \rightarrow F} dr \\ &- ((A - Y)_s(W) - (A - Y)_{\eta(s)}(W)) \xrightarrow{n \rightarrow \infty} 0. \end{aligned} \quad (3.107)$$

Moreover, with  $\tau_{k'_s} := \max\{k' : \tau_{k'} < s\}$  and  $d_s := s - \tau_{k'_s}$ , we have for sufficiently large  $n \in \mathbb{N}$

$$\begin{aligned} \sum_{I \in \mathcal{I}} Q(n; I) &= \int_{\eta(s)}^s \sum_{v \in (-\frac{1}{n}, \frac{1}{n})} \left\langle \Delta(A - Y)_{r-v}(W), g_n(v) \right\rangle_{F \rightarrow F} dr \\ &\leq \frac{C_5}{d_s} \left| \left( s - \frac{d_s}{2} \right) - \eta \left( s - \frac{d_s}{2} \right) \right| = \frac{C_5}{d_s} \left| \sigma \left( s - \frac{d_s}{2}, W_0 + (\bar{A} - \bar{Y})_{n, s - \frac{d_s}{2}} \right) - \left( s - \frac{d_s}{2} \right) \right| \end{aligned}$$

since we assume that  $s$  is not a jump time for  $X$ . Here  $C_5 \equiv C_5(S; W) < \infty$  is some positive constant not depending on  $n \in \mathbb{N}$ . The absolute values are taken coordinate wise. From Remark (1) (i), (iii) of Section 1 we get the existence of some positive constant  $C_6 \equiv C_6(S; W) < \infty$  not depending on  $n \in \mathbb{N}$  such that  $\sum_{I \in \mathcal{I}} Q(n; I) \leq C_6/d_s |(\bar{A} - \bar{Y})_{n, s - \frac{d_s}{2}}|$ .

Recalling how we have estimated in (3.31), by (3.93) there are constants  $C_7$  and  $C_8$  with the properties of  $C_6$  such that

$$\begin{aligned} \sum_{I \in \mathcal{I}} Q(n; I) &\leq \frac{C_6(1 + \alpha'((-s, s); W))}{d_s} |\psi(s - \frac{d_s}{2})| = \frac{C_7}{d_s} |\psi(s - \frac{d_s}{2})| \\ &= \frac{C_7}{d_s} \left| \hat{\psi}(\eta(s - \frac{d_s}{2})) \right| \leq \frac{C_8}{d_s} \int_{r=0}^{\eta(s)} \left\langle |\hat{\psi}(r)|, g_n(\eta(s - \frac{d_s}{2}) - r) \right\rangle_{F \rightarrow F} dr. \quad (3.108) \end{aligned}$$

For the last line we have also taken into consideration that because of (3.98) it holds that  $\eta(s - \frac{d_s}{2}) < \eta(s)$  for sufficiently large  $n \in \mathbb{N}$ .

Now we screen the terms  $R_1$ ,  $R_2$ ,  $R_3$ , and  $R_4$ . We take into consideration relations (3.97) as well as (3.98) and the fact that, in case that  $I \in \mathcal{I}$  is just a point, we have  $R_1 = R_2 = R_3 = R_4 = 0$ . We introduce

$$\hat{\beta}_1(r; n, W) := \alpha'(S; W) \cdot \mathbf{e} + 2\gamma'(S; W) \sum_{k \in \mathbb{Z} \setminus \{0\}} g_n(\tau_{k'} - r) + \frac{C_8}{d_s} g_n(\eta(s - \frac{d_s}{2}) - r),$$

which is the sum of the coefficients for  $|\hat{\psi}(r)|$  in  $Q$  and  $R_1$  as well as  $R_3$  for integration with respect to  $dr$ . In addition, using (3.97), we get some coefficient  $\hat{\beta}_2(r; n, W)$  for  $|\hat{\psi}(r)|$  coming from  $R_1$ ,  $R_3$ , and  $R_4$  for integration with respect to  $d\bar{\eta}^{-1}(r) - dr$ . Relations (3.104) and (3.105) give the idea how to construct  $\hat{\beta}_2(r; n, W)$ . We set  $\hat{\beta}(r; n, W) := \hat{\beta}_1(r; n, W) + \hat{\beta}_2(r; n, W)$ .

Furthermore, let  $\hat{R}(s; n, W)$  be the sum of  $\hat{\rho}_1(s; n, W)$ ,  $\hat{\rho}_2(s; n, W)$  (cf. (3.106) and (3.107)),  $\sum_{I \in \mathcal{I}} R_2(n; I)$ , and all other terms in  $\sum_{I \in \mathcal{I}} R_1(n; I)$ ,  $\sum_{I \in \mathcal{I}} R_3(n; I)$ ,  $\sum_{I \in \mathcal{I}} R_4(n; I)$  tending by (3.97) and (3.98) to zero as  $n \rightarrow \infty$ . Also here, relations (3.104) and (3.105) give the idea. We arrive at

$$\left| \hat{\psi}(\eta(s)) \right| = |\psi(s)| \leq \int_{r=0}^{\eta(s)} \left\langle |\hat{\psi}(r)|, \hat{\beta}(r; n, W) \right\rangle_{F \rightarrow F} dr + \hat{R}(s; n, W), \quad n \in \mathbb{N},$$

where all absolute values are coordinate wise. This is the counterpart to (3.33)-(3.35). The claim follows now as in (3.36).  $\square$

### 3.4 Proof of Theorem 1.11

**Proof of Theorem 1.11** Let  $W^{n,s}$ ,  $s \in \mathbb{R}$ , denote the flow defined in Lemma 3.4 (vi). In Step 1, we will establish an equation for the measures  $Q_\nu^{(m)} \circ W^{n,-v}$ ,  $v \in (0, t]$ , assuming  $t > 0$ .

Using this equation we will demonstrate in Step 2 that the Radon-Nikodym derivatives

$$\omega_{n,-v}^{(m)} := \frac{dQ_\nu^{(m)} \circ W^{n,-v}}{dQ_\nu^{(m)}}$$

exist in a certain sense. Furthermore, we will provide a representation of  $\omega_{n,-v}^{(m)}$ . In other words, we will derive a pre-version (1.1) on projections to  $H_i$ ,  $i \in J(m)$ , and  $e_j$ ,  $j \in \{1, \dots, n \cdot d\}$ , as far as jumps are not involved.

In Step 3, we are going to incorporate the jumps, i. e., we are going to determine the weak limit  $\lim_{n \rightarrow \infty} \omega_{n,-v}^{(m)} Q_\nu^{(m)}$ . In this way we will actually prove the existence of densities of the form

$$\omega_{-v}^{(m)} := \frac{dQ_\nu^{(m)} \circ W^{-v}}{dQ_\nu^{(m)}}, \quad v \in \mathbb{R}.$$

In Step 4 we will prepare the verification of the conditions of Theorem 2.1. Moreover, in Step 5, we shall verify (i) and (iii) of Theorem 2.1 and carry out the approximation  $\lim_{m \rightarrow \infty} \omega_{-t}^{(m)}$  which will lead to the representation (1.1). In Step 6, we are going to verify the remaining condition (ii) of Theorem 2.1 which will prove the theorem.

*Step 1* In this step, we shall derive an equation for the measures  $Q_\nu^{(m)} \circ W^{n,-v}$ ,  $v \in (0, t]$ .

Let  $\varphi$  be a cylindrical function on  $C(\mathbb{R}; F)$  of the form  $\varphi(W) = f_0(W_0) \cdot f_1(W_{t_1} - W_0, \dots, W_{t_l} - W_0)$  where  $f_0 \in C_b^1(F)$ ,  $f_1 \in C_b^1(F^l)$ , and  $t_i \in \{z \cdot t/2^m : z \in \mathbb{Z} \setminus \{0\}\}$ ,  $i \in \{1, \dots, l\}$ ,  $l \in \mathbb{N}$ . We notice that if we consider these test  $\varphi$  functions under the measure  $Q_\nu^{(m,r)}$  then we deal exactly with the test functions used in Lemma 3.2.

Let  $X_n = W + A_n$  and  $Y_n$ ,  $n \in \mathbb{N}$ , be the sequence of processes introduced in Lemma 3.4 and let  $W^{n,s} := W_{\cdot+s} + (A_{n,s} - Y_{n,s})\mathbb{1}$ ,  $s \in \mathbb{R}$ , be the flow which has been defined in Lemma 3.4 (vi) on  $\{(\pi_m W)_{\cdot+w} : W \in \Omega, w \in [0, \frac{1}{2^m} \cdot t]\}$ . For the sake of definiteness we extend this definition to  $\Omega$  by setting  $A_{n,\cdot}(W) := A(W)$  and  $Y_{n,\cdot}(W) := Y(W)$  if  $W \notin \{(\pi_m W)_{\cdot+w} : W \in \Omega, w \in [0, \frac{1}{2^m} \cdot t]\}$ . Note that now  $W^{n,s}$ ,  $s \in \mathbb{R}$ , is a flow on  $\Omega$  with  $W^{n,0} = W$ . We get

$$\begin{aligned} & \int \varphi \, dQ_\nu^{(m)} \circ W^{n,-v} - \int \varphi \, dQ_\nu^{(m)} \\ &= \int \varphi(W^{n,v}) Q_\nu^{(m)}(dW) - \int \varphi(W) Q_\nu^{(m)}(dW) \\ &= \int \left( \varphi(W_{\cdot+v} + (A_{n,v} - Y_{n,v})\mathbb{1}) - \varphi(W) \right) Q_\nu^{(m)}(dW) \\ &= \lim_{r \rightarrow \infty} \int \left( \varphi(W_{\cdot+v} + (A_{n,v} - Y_{n,v})\mathbb{1}) - \varphi(W) \right) Q_\nu^{(m,r)}(dW) \end{aligned} \quad (3.109)$$

where the limit  $\lim_{r \rightarrow \infty}$  follows from Lemma 3.4 (b) and Lemma 2.2 (c) with  $\psi \equiv \psi(A_n - Y_n)$ . For this limit we take also into consideration condition (4) (i) of Section 1 and that  $A_n = A_n^1$  and  $Y_n = Y_n^1$  by Lemma 3.4 (i).

Recalling Definition 2.6 (b) and using Lemma 3.2 (b), we obtain

$$\begin{aligned} & \int \left( \varphi(W_{\cdot+v} + (A_{n,v} - Y_{n,v})\mathbb{1}) - \varphi(W) \right) Q_\nu^{(m,r)}(dW) \\ &= \int \int_{\sigma=0}^v \left\langle (D\varphi)(\pi_{m,r}(W_{\cdot+\sigma} + (A_{n,\sigma} - Y_{n,\sigma})\mathbb{1})) \cdot, \right. \\ & \quad \left. j^{-1}(\pi_{m,r}(W_{\cdot+\sigma} + (A_{n,\sigma} - Y_{n,\sigma})\mathbb{1})) \cdot \right\rangle_H d\sigma Q_\nu^{(m,r)}(dW) \end{aligned}$$

where the “ $\cdot$ ” indicates the weak mixed derivative with respect to  $\sigma \in [0, t]$ . Because of Lemma 3.2 (c) this is thus equal to

$$\int \left( \varphi(W_{\cdot+v} + (A_{n,v} - Y_{n,v})\mathbb{1}) - \varphi(W) \right) Q_\nu^{(m,r)}(dW)$$

$$= \int \int_{\sigma=0}^v \left\langle D\varphi \circ (W_{\cdot+\sigma} + (A_{n,\sigma} - Y_{n,\sigma})\mathbb{I}) , \right. \\ \left. j^{-1} \left( \dot{W} + (\dot{A}_{n,0} - \dot{Y}_{n,0})\mathbb{I} \right) \right\rangle_H d\sigma Q_{\nu}^{(m,r)}(dW) .$$

Summarizing everything from (3.109) on, we get

$$\begin{aligned} & \int \varphi dQ_{\nu}^{(m)} \circ W^{n,-v} - \int \varphi dQ_{\nu}^{(m)} \\ &= \lim_{r \rightarrow \infty} \int \int_{\sigma=0}^v \left\langle D\varphi \circ (W_{\cdot+\sigma} + (A_{n,\sigma} - Y_{n,\sigma})\mathbb{I}) , \right. \\ & \quad \left. j^{-1} \left( \dot{W} + (\dot{A}_{n,0} - \dot{Y}_{n,0})\mathbb{I} \right) \right\rangle_H d\sigma Q_{\nu}^{(m,r)}(dW) \\ &= \lim_{r \rightarrow \infty} \int \int_{\sigma=0}^v \left\langle (D\varphi \circ (W_{\cdot+\sigma} + (A_{n,\sigma} - Y_{n,\sigma})\mathbb{I}))_1 , d\dot{W} \right\rangle_{L^2} d\sigma Q_{\nu}^{(m,r)}(dW) \\ & \quad + \lim_{r \rightarrow \infty} \int \int_{\sigma=0}^v \left\langle \nabla_{W_0} \varphi (W_{\cdot+\sigma} + (A_{n,\sigma} - Y_{n,\sigma})\mathbb{I}) , \dot{W} + \dot{A}_{n,0} - \dot{Y}_{n,0} \right\rangle_F d\sigma Q_{\nu}^{(m,r)}(dW) . \end{aligned} \quad (3.110)$$

In order to apply Lemma 3.2 (d) we note again that the weak mixed derivative defined in Subsection 2.3 is compatible with the integral of Definition 2.6 (a), cf. also Lemma 3.1 (b) and its proof. For the following this means in particular that

$$\left\langle (H_i, 0) , j^{-1} \left( \dot{W} + (\dot{A}_{n,0} - \dot{Y}_{n,0})\mathbb{I} \right) \right\rangle_H = \left\langle H_i, d\dot{W} \right\rangle_{L^2} .$$

By Lemma 3.2 (d)-(f) and by (2.3) as well as Lemma 3.3 we verify now

$$\begin{aligned} & \int \left\langle D\varphi \circ (W_{\cdot+\sigma} + (A_{n,\sigma} - Y_{n,\sigma})\mathbb{I}) , j^{-1} \left( \dot{W} + (\dot{A}_{n,0} - \dot{Y}_{n,0})\mathbb{I} \right) \right\rangle_H Q_{\nu}^{(m,r)}(dW) \\ &= \sum_{i \in I(m,r)} \int \left\langle D\varphi \circ (W_{\cdot+\sigma} + (A_{n,\sigma} - Y_{n,\sigma})\mathbb{I}) , (H_i, 0) \right\rangle_H \cdot \left\langle H_i, d\dot{W} \right\rangle_{L^2} Q_{\nu}^{(m,r)}(dW) \\ & \quad + \int \left\langle D\varphi \circ (W_{\cdot+\sigma} + (A_{n,\sigma} - Y_{n,\sigma})\mathbb{I}) , (0, \dot{W}_0 + \dot{A}_{n,0} - \dot{Y}_{n,0}) \right\rangle_H Q_{\nu}^{(m,r)}(dW) \\ &= \int \varphi (W_{\cdot+\sigma} + (A_{n,\sigma} - Y_{n,\sigma})\mathbb{I}) \times \\ & \quad \times \left( \sum_{i \in I(m,r)} \left( \left\langle H_i, d\dot{W} \right\rangle_{L^2} \cdot \delta(H_i, 0) - \left\langle D \left\langle H_i, d\dot{W} \right\rangle_{L^2} , (H_i, 0) \right\rangle_H \right) \right. \\ & \quad \left. - \left\langle \frac{\nabla m(W_0)}{m(W_0)} , \dot{W}_0 + \dot{A}_{n,0} - \dot{Y}_{n,0} \right\rangle_F - \left\langle \mathbf{e}, \nabla_{d,W_0} (\dot{A}_{n,0} - \dot{Y}_{n,0}) \right\rangle_F \right) Q_{\nu}^{(m,r)}(dW) \\ &= - \int \varphi (W_{\cdot+\sigma} + (A_{n,\sigma} - Y_{n,\sigma})\mathbb{I}) \left( \left\langle \frac{\nabla m(W_0)}{m(W_0)} , \dot{W}_0 + \dot{A}_{n,0} - \dot{Y}_{n,0} \right\rangle_F \right. \\ & \quad \left. + \left\langle \mathbf{e}, \nabla_{d,W_0} (\dot{A}_{n,0} - \dot{Y}_{n,0}) \right\rangle_F \right) Q_{\nu}^{(m,r)}(dW) \\ &\xrightarrow{r \rightarrow \infty} - \int \varphi (W_{\cdot+\sigma} + (A_{n,\sigma} - Y_{n,\sigma})\mathbb{I}) \left( \left\langle \frac{\nabla m(W_0)}{m(W_0)} , \dot{W}_0 + \dot{A}_{n,0} - \dot{Y}_{n,0} \right\rangle_F \right. \end{aligned}$$

$$+ \left\langle \mathbf{e}, \nabla_{d, W_0} \left( \dot{A}_{n,0} - \dot{Y}_{n,0} \right) \right\rangle_F \right) Q_\nu^{(m)}(dW). \quad (3.111)$$

Here the  $\lim_{r \rightarrow \infty}$  follows from Lemma 2.2 (c) with  $\psi \equiv \psi(A_n - Y_n, \dot{A}_n - \dot{Y}_n, \nabla_{d, W_0} \dot{A}_n - \nabla_{d, W_0} \dot{Y}_n)$  and Lemma 3.4 (b). In the following, let  $(\cdot)_c^{-1}$  denote the coordinate wise and inverse and  $\times_c$  denote the coordinate wise product. The log of a vector will denote the coordinate wise logarithm and the absolute value of a vector is going to be the vector of the absolute values. Recalling (3.109)-(3.111), it holds that

$$\begin{aligned} & \int \varphi \, dQ_\nu^{(m)} \circ W^{n,-v} - \int \varphi \, dQ_\nu^{(m)} \\ &= - \int_{\sigma=0}^v \int \varphi(W^{n,\sigma}) \left( \left\langle \frac{\nabla m(W_0)}{m(W_0)}, \dot{W}_0 + \dot{A}_{n,0} - \dot{Y}_{n,0} \right\rangle_F \right. \\ & \quad \left. + \left\langle \mathbf{e}, \nabla_{d, W_0} \left( \dot{A}_{n,0} - \dot{Y}_{n,0} \right) \right\rangle_F \right) Q_\nu^{(m)}(dW) \, d\sigma \\ &= - \int_{\sigma=0}^v \int \varphi \cdot \left( \left\langle \frac{\nabla m(W_0^{n,-\sigma})}{m(W_0^{n,-\sigma})}, \dot{X}_{n,-\sigma} - \dot{Y}_{n,-\sigma} \right\rangle_F \right. \\ & \quad \left. + \left\langle \mathbf{e}, \nabla_{d, W_0} \left( \dot{A}_{n,0} - \dot{Y}_{n,0} \right) (W^{n,-\sigma}) \right\rangle_F \right) dQ_\nu^{(m)} \circ W^{n,-\sigma} \, d\sigma \\ &= - \int_{\sigma=0}^v \int \varphi \cdot \left\langle \frac{\nabla m(W_0^{n,-\sigma})}{m(W_0^{n,-\sigma})}, \dot{X}_{n,-\sigma} - \dot{Y}_{n,-\sigma} \right\rangle_F dQ_\nu^{(m)} \circ W^{n,-\sigma} \, ds \\ & \quad - \int_{\sigma=0}^v \int \varphi \cdot \left\langle \mathbf{e}, \left( \nabla_{d, W_0} (A_{n,-\sigma} - Y_{n,-\sigma}) \right) \cdot \times_c \right. \\ & \quad \left. \times_c \left( \mathbf{e} + \nabla_{d, W_0} (A_{n,-\sigma} - Y_{n,-\sigma}) \right)_c^{-1} \right\rangle_F dQ_\nu^{(m)} \circ W^{n,-\sigma} \, d\sigma \\ &= - \int_{\sigma=0}^v \int \varphi \cdot \left\langle \frac{\nabla m(W_0^{n,-\sigma})}{m(W_0^{n,-\sigma})}, \dot{X}_{n,-\sigma} - \dot{Y}_{n,-\sigma} \right\rangle_F dQ_\nu^{(m)} \circ W^{n,-\sigma} \, ds \\ & \quad - \int_{\sigma=0}^v \int \varphi \cdot \left\langle \mathbf{e}, \left( \log \left| \mathbf{e} + \nabla_{d, W_0} (A_{n,-\sigma} - Y_{n,-\sigma}) \right| \right) \right\rangle_F dQ_\nu^{(m)} \circ W^{n,-\sigma} \, d\sigma \quad (3.112) \end{aligned}$$

where for the second equality sign we have applied the flow property of  $W^{n,\cdot}$  and in particular  $W^{n,0} = W$ . Furthermore, we have made use of  $X_{n,-\sigma}(W) = X_n(W^{n,-\sigma})$ ,  $Y_{n,-\sigma}(W) = Y_n(W^{n,-\sigma})$ , and  $\dot{A}_{n,-\sigma}(W) = \dot{A}_n(W^{n,-\sigma})$ , cf. Lemma 3.4 (vi).

*Step 2* Let  $\mathcal{F}^{(m)}$  denote the  $\sigma$ -algebra on  $\Omega$  which is generated by the projection operator  $\pi_m$ . In this step, we shall specify a subset of  $[0, t]$  such that, for  $v$  belonging to this subset, the Radon-Nikodym derivative  $dQ_\nu^{(m)} \circ W^{n,-v} / dQ_\nu^{(m)}$  exists on  $(\Omega, \mathcal{F}^{(m)})$ . We will also derive a representation of the density. Let us keep up with the following form of (3.112),

$$\begin{aligned} & \int \varphi \, dQ_\nu^{(m)} \circ W^{n,-v} - \int \varphi \, dQ_\nu^{(m)} \\ &= - \int_{\sigma=0}^v \int \varphi \cdot \left( \left\langle \frac{\nabla m(W_0)}{m(W_0)}, \dot{X}_{n,0} - \dot{Y}_{n,0} \right\rangle_F \right. \\ & \quad \left. + \left\langle \mathbf{e}, \nabla_{d, W_0} \left( \dot{A}_{n,0} - \dot{Y}_{n,0} \right) \right\rangle_F \right) dQ_\nu^{(m)} \circ W^{n,-\sigma} \, d\sigma. \end{aligned}$$

We recall that by  $\dot{X}_{n,0} - \dot{Y}_{n,0} = \dot{W} + \dot{A}_{n,0} - \dot{Y}_{n,0}$ , the definitions of the density  $m$  and the exponent  $q$  in Subsection 1.2, as well as by Lemma 3.4 (iii) and (iv),

$$\left( \left\langle \frac{\nabla m(W_0)}{m(W_0)}, \dot{X}_{n,0} - \dot{Y}_{n,0} \right\rangle_F + \left\langle \mathbf{e}, \nabla_{d,W_0} (\dot{A}_{n,0} - \dot{Y}_{n,0}) \right\rangle_F \right) dQ_\nu^{(m)}$$

is a finite signed measure. Using monotone convergence it turns out that (3.112) holds for all indicator functions  $\varphi(W) = \mathbb{I}_{\{W_{t_1} \in A_1, \dots, W_{t_l} \in A_l\}}$  where  $A_i \in \mathcal{B}(F)$  are open sets,  $t_i \in \{z \cdot t / 2^m : z \in \mathbb{Z}\}$ ,  $i \in \{1, \dots, l\}$ ,  $l \in \mathbb{N}$ . Using Sierpinski's monotone class theorem it follows now that (3.112) even holds for all  $\varphi \in B_b(\Omega, \mathcal{F}^{(m)})$ , the set of all bounded and with respect to  $\mathcal{F}^{(m)}$  measurable functions on  $\Omega$ .

Let us now establish a version of (3.112) for time dependent test functions  $\psi$  which are defined everywhere on  $[0, t] \times \Omega$  and satisfy the following.

$$\psi(v, \cdot) \in B_b(\Omega, \mathcal{F}^{(m)}), \quad v \in [0, t]. \quad (3.113)$$

There exists  $(\partial\psi/\partial v)(v, \cdot)$  such that

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{\psi(v+h, \cdot) \circ W^{n, v+h} - \psi(v, \cdot) \circ W^{n, v+h}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\psi(v+h, \cdot) \circ W^{n, v} - \psi(v, \cdot) \circ W^{n, v}}{h} \\ &= \frac{\partial\psi}{\partial v}(v, \cdot) \circ W^{n, v}, \quad v \in [0, t], \quad \text{in } L^1(\Omega, Q_\nu^{(m)}). \end{aligned} \quad (3.114)$$

It follows that, for all  $v \in [0, t]$ , there exists the limit

$$\begin{aligned} & \frac{d}{dv} \int \psi(v, \cdot) dQ_\nu^{(m)} \circ W^{n, -v} \\ &= \lim_{h \rightarrow 0} \frac{\int \psi(v+h, \cdot) dQ_\nu^{(m)} \circ W^{n, -v-h} - \int \psi(v, \cdot) dQ_\nu^{(m)} \circ W^{n, -v}}{h} \end{aligned}$$

whenever  $\psi$  satisfies (3.113) and (3.114) and that

$$\begin{aligned} & \frac{d}{dv} \int \psi(v, \cdot) dQ_\nu^{(m)} \circ W^{n, -v} - \int \frac{\partial\psi}{\partial v}(v, \cdot) dQ_\nu^{(m)} \circ W^{n, -v} \\ &= \frac{d}{dv} \int \psi(v, \cdot) dQ_\nu^{(m)} \circ W^{n, -v} - \lim_{h \rightarrow 0} \int \frac{\psi(v+h, \cdot) - \psi(v, \cdot)}{h} dQ_\nu^{(m)} \circ W^{n, -v} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \int \psi(v+h, \cdot) dQ_\nu^{(m)} \circ W^{n, -v-h} - \int \psi(v+h, \cdot) dQ_\nu^{(m)} \circ W^{n, -v} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \int \psi(v, \cdot) dQ_\nu^{(m)} \circ W^{n, -v-h} - \int \psi(v, \cdot) dQ_\nu^{(m)} \circ W^{n, -v} \right) \\ &\quad + \lim_{h \rightarrow 0} \int \frac{\psi(v+h, \cdot) - \psi(v, \cdot)}{h} dQ_\nu^{(m)} \circ W^{n, -v-h} \\ &\quad - \lim_{h \rightarrow 0} \int \frac{\psi(v+h, \cdot) - \psi(v, \cdot)}{h} dQ_\nu^{(m)} \circ W^{n, -v} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \int \psi(v, \cdot) dQ_\nu^{(m)} \circ W^{n, -v-h} - \int \psi(v, \cdot) dQ_\nu^{(m)} \circ W^{n, -v} \right) \end{aligned}$$

$$\begin{aligned}
&= - \int \psi(v, \cdot) \cdot \left( \left\langle \frac{\nabla m(W_0^{n,-v})}{m(W_0^{n,-v})}, \dot{X}_{n,-v} - \dot{Y}_{n,-v} \right\rangle_F \right. \\
&\quad \left. + \left\langle \mathbf{e}, \nabla_{d,W_0} \left( \dot{A}_{n,0} - \dot{Y}_{n,0} \right) (W^{n,-v}) \right\rangle_F \right) dQ_\nu^{(m)} \circ W^{n,-v}, \quad v \in [0, t],
\end{aligned}$$

the last line by (3.113) of the present step and (3.112). In addition, let us require that the subsequent condition on  $\psi$  is satisfied.

$$\int_0^t \left| \frac{\partial \psi}{\partial v}(v, \cdot) \circ W^{n,v} \right| dv \in L^1(\Omega, Q_\nu^{(m)}). \quad (3.115)$$

Now the last equation has an integral version

$$\begin{aligned}
&\int \psi(v, \cdot) dQ_\nu^{(m)} \circ W^{n,-v} - \int \psi(0, \cdot) dQ_\nu^{(m)} - \int_{\sigma=0}^v \int \frac{\partial \psi}{\partial \sigma}(\sigma, \cdot) dQ_\nu^{(m)} \circ W^{n,-\sigma} d\sigma \\
&= - \int_{\sigma=0}^v \int \psi(\sigma, \cdot) \cdot \left( \left\langle \frac{\nabla m(W_0^{n,-\sigma})}{m(W_0^{n,-\sigma})}, \dot{X}_{n,-\sigma} - \dot{Y}_{n,-\sigma} \right\rangle_F \right. \\
&\quad \left. + \left\langle \mathbf{e}, \nabla_{d,W_0} \left( \dot{A}_{n,0} - \dot{Y}_{n,0} \right) (W^{n,-\sigma}) \right\rangle_F \right) dQ_\nu^{(m)} \circ W^{n,-\sigma} d\sigma, \quad (3.116)
\end{aligned}$$

$v \in [0, t]$ . Let  $\Psi(v, \cdot) \in B_b(\Omega, \mathcal{F}^{(m)})$ ,  $v \in [0, t]$ . Until further specification, assume that the function

$$\hat{\psi}(v, \cdot) := \exp \left\{ \int_{\sigma=0}^v \Psi(\sigma, \cdot) d\sigma \right\} \cdot \mathbb{1}_G, \quad v \in [0, t], \quad (3.117)$$

is for any  $G \in \mathcal{F}^{(m)}$  and  $c > 0$  a function satisfying (3.113)-(3.115). Thus  $\hat{\psi}$  satisfies also (3.116). We obtain from (3.116)

$$\begin{aligned}
&\int \hat{\psi}(v, \cdot) dQ_\nu^{(m)} \circ W^{n,-v} - Q_\nu^{(m)}(G) \\
&= \int \hat{\psi}(v, \cdot) dQ_\nu^{(m)} \circ W^{n,-v} - \int \hat{\psi}(0, \cdot) dQ_\nu^{(m)} \\
&= \int_{\sigma=0}^v \int \hat{\psi}(\sigma, \cdot) \cdot \left( \Psi(\sigma, \cdot) - \left\langle \frac{\nabla m(W_0^{n,-\sigma})}{m(W_0^{n,-\sigma})}, \dot{X}_{n,-\sigma} - \dot{Y}_{n,-\sigma} \right\rangle_F \right. \\
&\quad \left. - \left\langle \mathbf{e}, \nabla_{d,W_0} \left( \dot{A}_{n,0} - \dot{Y}_{n,0} \right) (W^{n,-\sigma}) \right\rangle_F \right) dQ_\nu^{(m)} \circ W^{n,-\sigma} d\sigma, \quad v \in [0, t]. \quad (3.118)
\end{aligned}$$

The following observation is a consequence of the law of iterated logarithm and elementary geometric properties of piecewise linear functions with equidistant linearity intervals. For every  $v \in [0, t] \setminus \{z \cdot t/2^m + t/2^{m+1} : z \in \mathbb{Z}\}$  and every  $W \in \Omega$  with

$$\lim_{\substack{z \rightarrow \pm\infty \\ z \in \mathbb{Z}}} \frac{1}{z} \log |W_{z \cdot t/2^m}| = 0$$

there exists exactly one  $V \in \{(\pi_m U)_{\cdot+v} : U \in \Omega\}$  with  $V_s = W_s$  for all  $s \in \{z \cdot t/2^m : z \in \mathbb{Z}\}$ , which also means that  $\pi_m W = \pi_m V$ , such that

$$\lim_{\substack{z \rightarrow \pm\infty \\ z \in \mathbb{Z}}} \frac{1}{z} \log |V_{z \cdot t/2^m + v}| = 0.$$

In this notation we shall write  $V = \pi_{m;v}W$ . Recalling  $W^{n,-v} = W_{-v} + (A_{-v} - Y_{-v})\mathbb{1}$  it is obvious that

$$Q_\nu^{(m)} \circ W^{n,-v} \left( V \in \{\pi_{m;v}W : W \in \Omega\} : \lim_{\substack{z \rightarrow \pm\infty \\ z \in \mathbb{Z}}} \frac{1}{z} \log |V_{z \cdot t/2^m + v}| = 0 \right) = 1, \quad v \in [0, t].$$

In other words, with  $E_\nu^{(m)} \circ W^{n,-v}$  denoting the expectation with respect to the measure  $Q_\nu^{(m)} \circ W^{n,-v}$ ,  $v \in [0, t]$ , and  $\mathcal{F}^{(m)} \subset \mathcal{F}$  being the  $\sigma$ -algebra generated by  $\pi_m$ , it holds that

$$E_\nu^{(m)} \circ W^{n,-v} (\xi | \mathcal{F}^{(m)}) = \xi \circ \pi_{m;v}, \quad \xi \in L^1(\Omega, Q_\nu^{(m)} \circ W^{n,-v}), \quad (3.119)$$

$v \in [0, t] \setminus \{z \cdot t/2^m + t/2^{m+1} : z \in \mathbb{Z}\}$ . Furthermore, for the same  $v$  we have

$$\xi \circ \pi_{m;v} = \xi, \quad Q_\nu^{(m)} \circ W^{n,-v} \text{ -a.e.} \quad (3.120)$$

Next, let us specify  $\hat{\psi}$  in (3.117). For  $c > 0$  we choose

$$\Phi_c(v, \cdot) = \left( \left\langle \frac{\nabla m(W_0^{n,-v})}{m(W_0^{n,-v})}, \dot{X}_{n,-v} - \dot{Y}_{n,-v} \right\rangle_F + \left\langle \mathbf{e}, \nabla_{d,W_0} (\dot{A}_{n,0} - \dot{Y}_{n,0}) (W^{n,-v}) \right\rangle_F \right) \wedge c \quad (3.121)$$

and define

$$\Psi_c(v, \cdot) := \Phi_c(v, \cdot) \circ \pi_{m;v} \quad (3.122)$$

as well as

$$\hat{\psi}_c(v, \cdot) := \exp \left\{ \int_{\sigma=0}^v \Psi_c(\sigma, \cdot) d\sigma \right\} \cdot \mathbb{1}_G,$$

$v \in [0, t] \setminus \{z \cdot t/2^m + t/2^{m+1} : z \in \mathbb{Z}\}$ . By (3.119) we see that  $\hat{\psi}_c$  has in particular the form of  $\hat{\psi}$  in (3.117). We observe that

$$\hat{\psi}_c(v, \cdot) = \exp \left\{ \int_{\sigma=0}^v \Phi_c(\sigma, \cdot) d\sigma \right\} \circ \pi_{m;v} \cdot \mathbb{1}_G, \quad (3.123)$$

$v \in [0, t] \setminus \{z \cdot t/2^m + t/2^{m+1} : z \in \mathbb{Z}\}$ . In order to apply (3.118) to  $\hat{\psi}_c$  we have to verify (3.113)-(3.115). Relation (3.113) is clear by definition. Relation (3.114) is on the one hand a consequence of the fact that for  $k(h)$  either  $h$  or  $0$  we have

$$\begin{aligned} & \left| \frac{\hat{\psi}_c(v+h, \cdot) \circ W^{n,v+k(h)} - \hat{\psi}_c(v, \cdot) \circ W^{n,v+k(h)}}{h} \right| \\ &= \left| \frac{1}{h} \int_{a=0}^h \Psi_c(v+a, W^{n,v+k(a)}) \cdot \exp \left\{ \int_{\sigma=0}^{v+a} \Psi_c(\sigma, W^{n,v+k(a)}) d\sigma \right\} da \cdot \mathbb{1}_G \right| \\ &\xrightarrow{h \rightarrow 0} \Psi_c(v, W^{n,v}) \cdot \exp \left\{ \int_{\sigma=0}^v \Psi_c(\sigma, W^{n,v}) d\sigma \right\} \cdot \mathbb{1}_G \quad Q_\nu^{(m)} \text{-a.e.} \end{aligned}$$

by (3.121) and (3.122) as well as Lemma 3.4 (i), (v), and (vi). On the other hand,

$$\left\| \Psi_c(v+a, W^{n,v+k(a)}) \cdot \exp \left\{ \int_{\sigma=0}^{v+a} \Psi_c(\sigma, W^{n,v+k(a)}) d\sigma \right\} \right\|_{L^1(\Omega, Q_\nu^{(m)})} \leq c \cdot e^{c(v+\varepsilon)}$$

independent of  $a \in (-\varepsilon, \varepsilon)$  for some  $\varepsilon > 0$ ,  $v \in [0, t] \setminus \{z \cdot t/2^m + t/2^{m+1} : z \in \mathbb{Z}\}$ , which completes the verification of (3.114). The last relation yields also (3.115) for  $\hat{\psi}_c$ .

It follows now from (3.118) as well as (3.120) and (3.123) that

$$\begin{aligned} & \int_G \exp \left\{ \int_{\sigma=0}^v \Phi_c(\sigma, \cdot) d\sigma \right\} dQ_\nu^{(m)} \circ W^{n,-v} - Q_\nu^{(m)}(G) \\ &= \int \hat{\psi}_c(v, \cdot) dQ_\nu^{(m)} \circ W^{n,-v} - Q_\nu^{(m)}(G) \\ &= \int_{\sigma=0}^v \int \hat{\psi}_c(\sigma, \cdot) \cdot \left( \Psi_c(\sigma, \cdot) - \left\langle \frac{\nabla m(W_0^{n,-\sigma})}{m(W_0^{n,-\sigma})}, \dot{X}_{n,-\sigma} - \dot{Y}_{n,-\sigma} \right\rangle_F \right. \\ & \quad \left. - \left\langle \mathbf{e}, \nabla_{d,W_0} (\dot{A}_{n,0} - \dot{Y}_{n,0}) (W^{n,-\sigma}) \right\rangle_F \right) dQ_\nu^{(m)} \circ W^{n,-\sigma} d\sigma \\ &= \int_{\sigma=0}^v \int \hat{\psi}_c(\sigma, \cdot) \cdot \left( \Phi_c(\sigma, \cdot) - \left\langle \frac{\nabla m(W_0^{n,-\sigma})}{m(W_0^{n,-\sigma})}, \dot{X}_{n,-\sigma} - \dot{Y}_{n,-\sigma} \right\rangle_F \right. \\ & \quad \left. - \left\langle \mathbf{e}, \nabla_{d,W_0} (\dot{A}_{n,0} - \dot{Y}_{n,0}) (W^{n,-\sigma}) \right\rangle_F \right) dQ_\nu^{(m)} \circ W^{n,-\sigma} d\sigma, \quad (3.124) \end{aligned}$$

$v \in [0, t] \setminus \{z \cdot t/2^m + t/2^{m+1} : z \in \mathbb{Z}\}$ . In particular, (3.124) together with (3.121) means that

$$\int_G \exp \left\{ \int_{\sigma=0}^v \Phi_c(\sigma, \cdot) d\sigma \right\} dQ_\nu^{(m)} \circ W^{n,-v} \leq Q_\nu^{(m)}(G), \quad G \in \mathcal{F}^{(m)},$$

$v \in [0, t] \setminus \{z \cdot t/2^m + t/2^{m+1} : z \in \mathbb{Z}\}$ . Extending definition (3.121) to  $c = \infty$ , monotone convergence yields for the same  $v$

$$\int_G \exp \left\{ \int_{\sigma=0}^v \Phi_\infty(\sigma, \cdot) d\sigma \right\} dQ_\nu^{(m)} \circ W^{n,-v} \leq Q_\nu^{(m)}(G), \quad G \in \mathcal{F}^{(m)}. \quad (3.125)$$

Plugging (3.123) into (3.124), taking into consideration (3.120), and plugging then (3.125) into (3.124) we obtain

$$\begin{aligned} 0 &\leq Q_\nu^{(m)}(G) - \int_G \exp \left\{ \int_{\sigma=0}^v \Phi_c(\sigma, \cdot) d\sigma \right\} dQ_\nu^{(m)} \circ W^{n,-v} \\ &= \int_{\sigma=0}^v \int_G \exp \left\{ \int_{u=0}^\sigma \Phi_c(u, \cdot) du \right\} \cdot \left( \Phi_\infty(\sigma, \cdot) - \Phi_c(\sigma, \cdot) \right) dQ_\nu^{(m)} \circ W^{n,-\sigma} d\sigma \\ &\leq \int_{\sigma=0}^v \int_G \exp \left\{ \int_{u=0}^\sigma \left( (\Phi_\infty(u, \cdot) - \Phi_c(u, \cdot)) \vee c \right) du \right\} \times \\ & \quad \times \left( (\Phi_\infty(\sigma, \cdot) - \Phi_c(\sigma, \cdot)) \vee c \right) dQ_\nu^{(m)} d\sigma \\ &= \int_G \left( \exp \left\{ \int_{\sigma=0}^v \left( (\Phi_\infty(\sigma, \cdot) - \Phi_c(\sigma, \cdot)) \vee c \right) d\sigma \right\} - 1 \right) dQ_\nu^{(m)}, \quad G \in \mathcal{F}^{(m)}, \quad (3.126) \end{aligned}$$

$v \in [0, t] \setminus \{z \cdot t/2^m + t/2^{m+1} : z \in \mathbb{Z}\}$ . Letting  $c \rightarrow \infty$  in (3.126) and recalling (3.125), it follows again by monotone convergence that for such  $v$

$$\int_G \exp \left\{ \int_{\sigma=0}^v \Phi_\infty(\sigma, \cdot) d\sigma \right\} dQ_\nu^{(m)} \circ W^{n,-v} = Q_\nu^{(m)}(G), \quad G \in \mathcal{F}^{(m)}.$$

Noting that

$$\begin{aligned} & - \int_{\sigma=0}^v \left\langle \frac{\nabla m(X_{n,-\sigma} - Y_{n,-\sigma})}{m(X_{n,-\sigma} - Y_{n,-\sigma})}, d(X_{n,-\sigma} - Y_{n,-\sigma}) \right\rangle_F \\ & = \log m(X_{n,v} - Y_{n,v}) - \log m(X_{n,0} - Y_{n,0}) \end{aligned}$$

is path wise just a Lebesgue-Stieltjes integral with no jumps in the integrator and recalling  $X_{n,0} - Y_{n,0} = W_0 + A_{n,0} - Y_{n,0} = W_0$  and Lemma 3.4 (v), we have shown that on  $(\Omega, \mathcal{F}^{(m)})$  the Radon-Nikodym derivative  $dQ_\nu^{(m)} \circ W^{n,-v} / dQ_\nu^{(m)}$  exists and that it coincides with

$$\begin{aligned} \omega_{n,-v}^{(m)} &:= \frac{dQ_\nu^{(m)} \circ W^{n,-v}}{dQ_\nu^{(m)}} = \exp \left\{ - \int_{\sigma=0}^v \Phi_\infty(\sigma, \cdot) d\sigma \right\} \\ &= \exp \left\{ - \int_{\sigma=0}^v \left\langle \frac{\nabla m(X_{n,-\sigma} - Y_{n,-\sigma})}{m(X_{n,-\sigma} - Y_{n,-\sigma})}, \dot{X}_{n,-\sigma} - \dot{Y}_{n,-\sigma} \right\rangle_F d\sigma \right\} \times \\ &\quad \times \exp \left\{ - \int_{\sigma=0}^v \left\langle \mathbf{e}, \nabla_{d,W_0} (\dot{A}_{n,0} - \dot{Y}_{n,0}) (W^{n,-\sigma}) \right\rangle_F d\sigma \right\} \\ &= \frac{m(X_{n,-v} - Y_{n,-v})}{m(W_0)} \cdot \exp \left\{ - \int_{\sigma=0}^v \left\langle \mathbf{e}, \left( \log \left| \mathbf{e} + \nabla_{d,W_0} (A_{n,-\sigma} - Y_{n,-\sigma}) \right| \right) \right\rangle_F d\sigma \right\} \\ &= \frac{m(X_{n,-v} - Y_{n,-v})}{m(W_0)} \cdot \exp \left\{ \left\langle \mathbf{e}, \log \left| \mathbf{e} + \nabla_{d,W_0} (A_{n,-v} - Y_{n,-v}) \right| \right\rangle_F \right\} \\ &= \frac{m(X_{n,-v} - Y_{n,-v})}{m(W_0)} \cdot \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d,W_0} (A_{n,-v} - Y_{n,-v}) \right|_i \end{aligned}$$

$Q_\nu^{(m)}$ -a.e.,  $v \in [0, t] \setminus \{z \cdot t/2^m + t/2^{m+1} : z \in \mathbb{Z}\}$ . Recall also (3.112).

*Step 3* Now we are interested in the weak limit  $\lim_{n \rightarrow \infty} \omega_{n,-v}^{(m)} Q_\nu^{(m)}$  on  $(\Omega, \mathcal{F}^{(m)})$ ,  $v \in [0, t] \setminus \{z \cdot t/2^m + t/2^{m+1} : z \in \mathbb{Z}\}$ . By Lemma 3.4 (iv), (c), and (a) together with the fact that  $Y, X$  jump for fixed  $v$  only on a  $Q_\nu^{(m)}$ -zero set, cf. conditions (2) and (3) of Section 1, the boundedness and continuity of the density  $m$ , and again  $A_{n,0} = Y_{n,0}$ , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \omega_{n,-v}^{(m)} &= \lim_{n \rightarrow \infty} \frac{m(X_{n,-v} - Y_{n,-v})}{m(W_0)} \cdot \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d,W_0} (A_{n,-v} - Y_{n,-v}) \right|_i \\ &= \frac{m(X_{-v} - Y_{-v})}{m(W_0)} \cdot \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d,W_0} A_{-v} - \nabla_{d,W_0} Y_{-v} \right|_i \end{aligned} \quad (3.127)$$

$Q_\nu^{(m)}$ -a.e. on  $(\Omega, \mathcal{F}^{(m)})$ ; recall also Remark (3) of this section. Let us turn to test functions  $\varphi$  of the form  $\varphi(W) = f(W_{t_1}, \dots, W_{t_l})$  where  $f \in C_b(F^l)$ , and  $t_i \in \{z \cdot t/2^m : z \in \mathbb{Z}\}$ ,  $i \in \{1, \dots, l\}$ ,  $l \in \mathbb{N}$ . For  $v \in [0, t] \setminus \{z \cdot t/2^m + t/2^{m+1} : z \in \mathbb{Z}\}$  we have

$$\int \varphi dQ_\nu^{(m)} \circ W^{-v}$$

$$\begin{aligned}
&= \int \varphi(W^v) Q_\nu^{(m)}(dW) \\
&= \int \varphi(W_{\cdot+v} + (A_v - Y_v)\mathbb{1}) Q_\nu^{(m)}(dW) \\
&= \lim_{n \rightarrow \infty} \int \varphi(W_{\cdot+v} + (A_{n,v} - Y_{n,v})\mathbb{1}) Q_\nu^{(m)}(dW) \\
&= \lim_{n \rightarrow \infty} \int \varphi dQ_\nu^{(m)} \circ W^{n,-v} \\
&= \lim_{n \rightarrow \infty} \int \varphi \omega_{n,-v}^{(m)} dQ_\nu^{(m)} \\
&= \lim_{n \rightarrow \infty} \int \varphi m(X_{n,-v} - Y_{n,-v}) \cdot \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d,W_0} A_{n,-v} - \nabla_{d,W_0} Y_{n,-v} \right|_i dQ_{\lambda_F}^{(m)} \\
&= \int \varphi \frac{m(X_{-v} - Y_{-v})}{m(W_0)} \cdot \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d,W_0} A_{-v} - \nabla_{d,W_0} Y_{-v} \right|_i dQ_\nu^{(m)}
\end{aligned}$$

where the limit  $\lim_{n \rightarrow \infty}$  in the fourth line is a consequence of Lemma 3.4 (a) together with the fact that  $A, Y$  jump for fixed  $v$  only on a  $Q_\nu^{(m)}$ -zero set, cf. conditions (2) and (3) of Section 1. The last line follows from (3.127) and Lemma 3.4 (c). For  $v \in [0, t] \setminus \{z \cdot t/2^m + t/2^{m+1} : z \in \mathbb{Z}\}$  we have shown

$$\omega_{-v}^{(m)}(W) := \frac{dQ_\nu^{(m)} \circ W^{-v}}{dQ_\nu^{(m)}} = \frac{m(X_{-v} - Y_{-v})}{m(W_0)} \cdot \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d,W_0} A_{-v} - \nabla_{d,W_0} Y_{-v} \right|_i \quad (3.128)$$

on  $(\Omega, \mathcal{F}^{(m)})$ . In the remainder of this step we show that (3.128) holds also for negative  $v$ . To do so we still let  $v \geq 0$  and take a look on  $\omega_{-v}(W^v)$ . We also recall Lemma 3.4 (vi), Lemma 3.4 (a) and (c), as well as conditions 2 (i) and (4) (i) of Section 1 and obtain  $Q_\nu^{(m)}$ -a.e.

$$\begin{aligned}
\omega_{-v}^{(m)}(W^v) &= \frac{m(X_{-v}(W^v) - Y_{-v}(W^v))}{m(W_0^v)} \cdot \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d,W_0} A_{-v}(W^v) - \nabla_{d,W_0} Y_{-v}(W^v) \right|_i \\
&= \frac{m(W_0)}{m(W_0^v)} \cdot \exp \left\{ \left\langle \mathbf{e}, \log \left| \mathbf{e} + \nabla_{d,W_0} A_{-v}(W^v) - \nabla_{d,W_0} Y_{-v}(W^v) \right| \right\rangle_F \right\} \\
&= \lim_{n \rightarrow \infty} \left( \frac{m(W_0)}{m(W_0^{n,v})} \cdot \exp \left\{ \left\langle \mathbf{e}, \log \left| \mathbf{e} + \nabla_{d,W_0} A_{n,-v}(W^{n,v}) - \nabla_{d,W_0} Y_{n,-v}(W^{n,v}) \right| \right\rangle_F \right\} \right) \\
&= \lim_{n \rightarrow \infty} \left( \frac{m(W_0)}{m(W_0^{n,v})} \cdot \exp \left\{ - \int_{s=0}^v \left\langle \mathbf{e}, \left( \log \left| \mathbf{e} + \nabla_{d,W_0} (A_{n,-s} - Y_{n,-s}) \right| \right) \right\rangle \times_c \right. \right. \\
&\quad \left. \left. \times_c (W^{n,v}) \right\rangle_F ds \right\} \right) \\
&= \lim_{n \rightarrow \infty} \left( \frac{m(W_0)}{m(W_0^{n,v})} \cdot \exp \left\{ - \int_{s=0}^v \left\langle \mathbf{e}, \nabla_{d,W_0} \left( (\dot{A}_{n,0} - \dot{Y}_{n,0})(W^{n,-s}) \right) \times_c \right. \right. \right. \\
&\quad \left. \left. \times_c \left( \mathbf{e} + \nabla_{d,W_0} (A_{n,-s} - Y_{n,-s}) \right)_c^{-1} \right\rangle_F ds \circ W^{n,v} \right\} \right) \\
&= \lim_{n \rightarrow \infty} \left( \frac{m(W_0)}{m(W_0^{n,v})} \cdot \exp \left\{ - \int_{s=0}^v \left\langle \mathbf{e}, \nabla_{d,W_0} \left( \dot{A}_{n,0} - \dot{Y}_{n,0} \right)(W^{n,-s}) \right\rangle_F ds \circ W^{n,v} \right\} \right)
\end{aligned}$$

$$= \lim_{n \rightarrow \infty} \left( \frac{m(W_0)}{m(W_0^{n,v})} \cdot \exp \left\{ - \int_{s=-v}^0 \left\langle \mathbf{e}, \nabla_{d,W_0} \left( \dot{A}_{n,0} - \dot{Y}_{n,0} \right) (W^{n,-s}) \right\rangle_F ds \right\} \right) .$$

Performing similar calculations, now the other way around, we obtain

$$\begin{aligned} \frac{dQ_\nu^{(m)} \circ W^v}{dQ_\nu^{(m)}} &= \frac{1}{\omega_{-v}^{(m)}(W^v)} \\ &= \lim_{n \rightarrow \infty} \left( \frac{m(W_0^{n,v})}{m(W_0)} \cdot \exp \left\{ - \int_{s=0}^{-v} \left\langle \mathbf{e}, \nabla_{d,W_0} \left( \dot{A}_{n,0} - \dot{Y}_{n,0} \right) (W^{n,-s}) \right\rangle_F ds \right\} \right) \\ &= \lim_{n \rightarrow \infty} \left( \frac{m(W_0^{n,v})}{m(W_0)} \cdot \exp \left\{ - \int_{s=0}^{-v} \left\langle \mathbf{e}, \left( \log \left| \mathbf{e} + \nabla_{d,W_0} (A_{n,-s} - Y_{n,-s}) \right| \right) \right\rangle_F ds \right\} \right) \\ &= \lim_{n \rightarrow \infty} \left( \frac{m(W_0^{n,v})}{m(W_0)} \cdot \exp \left\{ \left\langle \mathbf{e}, \log \left| \mathbf{e} + \nabla_{d,W_0} A_{n,v} - \nabla_{d,W_0} Y_{n,v} \right| \right\rangle_F \right\} \right) \\ &= \frac{m(W_0^v)}{m(W_0)} \cdot \exp \left\{ \left\langle \mathbf{e}, \log \left| \mathbf{e} + \nabla_{d,W_0} A_v - \nabla_{d,W_0} Y_v \right| \right\rangle_F \right\} \\ &= \frac{m(X_v - Y_v)}{m(W_0)} \cdot \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d,W_0} A_v - \nabla_{d,W_0} Y_v \right|_i . \end{aligned}$$

This implies that relation (3.128) holds also for  $v < 0$ .

*Step 4* In this step we will prepare the verification of the conditions of Theorem 2.1.

Let us consider the Lévy-Ciesielsky construction of the process  $W_s = \eta + \sum_{i=1}^{\infty} \xi_i \cdot \int_0^s H_i(u) du$ ,  $s \in \mathbb{R}$ , with independent  $N(0, t)$ -distributed random variables  $\xi_1, \xi_2, \dots$  and  $\eta$  being an  $F$ -valued random variable with distribution  $\nu$  independent of  $\xi_1, \xi_2, \dots$ . We recall that the right-hand side converges  $Q_\nu$ -a.e. uniformly in  $s$  on compact subsets of  $\mathbb{R}$  and that  $\xi_i = \langle H_i, dW \rangle_{L^2}$ ,  $i \in \mathbb{N}$ , and  $\eta = W_0$ . In Subsection 2.1 we had introduced the projection of  $W$  to the linear span of

$$\left\{ \int_0^\cdot H_i(u) du, e_j \cdot \mathbb{1} : i \in J(m), j \in \{1, \dots, n \cdot d\} \right\}$$

by considering the process  $W$  under the measure  $Q_\nu^{(m)}$ .

For a path  $W \in \Omega$  with  $W_s = y + \sum_{i=1}^{\infty} x_i \cdot \int_0^s H_i(u) du$ ,  $s \in \mathbb{R}$ , we will use the identifications

$$Q_\nu(dW) \equiv Q_\nu(\eta \in dy, \{\xi_i \in dx_i : i \in \mathbb{N}\})$$

and for

$$W_s^{(m)} := \eta + \sum_{i \in J(m)} \xi_i \cdot \int_0^s H_i(u) du, \quad s \in \mathbb{R},$$

we shall write

$$Q_\nu^{(m)}(dW^{(m)}) \equiv Q_\nu(\eta \in dy, \{\xi_i \in dx_i : i \in J(m)\}) .$$

Define  $A^{(m)} := A(W^{(m)})$  and  $Y^{(m)} := Y(W^{(m)})$ . Let  $i \in \mathbb{N}$  and  $i' \equiv i'(i)$  are related by  $H_{i'} = H_i(\cdot + t)$ . Note that  $i \rightarrow i'$  is a bijection of type  $\mathbb{N} \rightarrow \mathbb{N}$  or  $J(m) \rightarrow J(m)$ . It follows from the Lévy-Ciesielski construction of the path  $W \in \Omega$  that

$$\begin{aligned} Z_t(W) &:= W_{\cdot-t} + \left( A_{-t}^{(m)} - Y_{-t}^{(m)} \right) \mathbb{1} \\ &= \left( \eta + A_{-t}^{(m)} - Y_{-t}^{(m)} \right) \mathbb{1} + \sum_{i \in \mathbb{N}} x_i \cdot \int_0^{\cdot-t} H_i(u) du \\ &= \left( \eta + A_{-t}^{(m)} - Y_{-t}^{(m)} \right) \mathbb{1} + \sum_{i \in \mathbb{N}} x_{i'} \cdot \int_t^{\cdot} H_i(u) du. \end{aligned} \quad (3.129)$$

Similarly, we obtain

$$W_{\cdot-t}^{(m)} + \left( A_{-t}^{(m)} - Y_{-t}^{(m)} \right) \mathbb{1} = \left( \eta + A_{-t}^{(m)} - Y_{-t}^{(m)} \right) \mathbb{1} + \sum_{i \in J(m)} x_{i'} \cdot \int_t^{\cdot} H_i(u) du. \quad (3.130)$$

Let us demonstrate that (3.129) implies that  $W \rightarrow W_{\cdot-t} + \left( A_{-t}^{(m)} - Y_{-t}^{(m)} \right) \mathbb{1}$  is an injection with

$$Q_\nu \left( \left\{ W_{\cdot-t} + \left( A_{-t}^{(m)} - Y_{-t}^{(m)} \right) \mathbb{1} : W \in \Omega \right\} \right) = 1. \quad (3.131)$$

To begin with let us verify injectivity. Given  $Z$ , the values of  $x_i$ ,  $i \in \mathbb{N}$ , are uniquely determined by the linear combination  $Z - Z_0 \mathbb{1} = \sum_{i \in \mathbb{N}} x_{i'} \cdot \int_0^{\cdot} H_i(u) du$  and the bijectivity of  $i \rightarrow i'$ . It remains to demonstrate that for given  $x_i$ ,  $i \in \mathbb{N}$ , and  $\eta + A_{-t}^{(m)} - Y_{-t}^{(m)}$ , the value of  $\eta$  is unique. Because of (3.130) the term  $W_{\cdot-t}^{(m)} + \left( A_{-t}^{(m)} - Y_{-t}^{(m)} \right) \mathbb{1}$  is now given and therefore

$$X_{\cdot-t}(W^{(m)}) = u \left( W_{\cdot-t}^{(m)} + \left( A_{-t}^{(m)} - Y_{-t}^{(m)} \right) \mathbb{1} \right) = (W^{(m)})^{-t} + A \left( (W^{(m)})^{-t} \right)$$

is given as well, cf. condition (3) of Section 1. By the injectivity of the map  $u$ , cf. Subsection 1.2, we get  $Y_{-t}^{(m)} = Y_{-t}(W^{(m)}) = A_0 \left( (W^{(m)})^{-t} \right)$  uniquely. Furthermore, we get  $X(W^{(m)}) = W^{(m)} + A^{(m)}$  and again by the injectivity of the map  $u$  this yields  $A_{-t}^{(m)}$  uniquely. This shows finally that  $\eta$  is unique. Therefore,  $W \rightarrow W_{\cdot-t} + \left( A_{-t}^{(m)} - Y_{-t}^{(m)} \right) \mathbb{1}$  is an injection. From  $W_{\cdot-t}^{(m)} = (W_{\cdot-t})^{(m)}$  and  $\varphi \equiv 1$  in (3.110) we obtain

$$Q_\nu^{(m)} \left( \left\{ W_{\cdot-t}^{(m)} + \left( A_{-t}^{(m)} - Y_{-t}^{(m)} \right) \mathbb{1} : W \in \Omega \right\} \right) = 1.$$

Together with (3.129), (3.130), and the independence of  $A^{(m)}$  of  $\xi_i$ ,  $i \notin J(m)$ , this means

$$\begin{aligned} 1 &= Q_\nu^{(m)} \left( \left\{ W_{\cdot-t}^{(m)} + \left( A_{-t}^{(m)} - Y_{-t}^{(m)} \right) \mathbb{1} : W \in \Omega \right\} \right) \cdot Q_\nu(\{\xi_i \in \mathbb{R} : i \notin J(m)\}) \\ &= Q_\nu \left( \left\{ W_{\cdot-t} + \left( A_{-t}^{(m)} - Y_{-t}^{(m)} \right) \mathbb{1} : W \in \Omega \right\} \right) \end{aligned}$$

which is (3.131). The flow property of  $W^v$ ,  $v \in \mathbb{R}$ , verified in Remark (3) of Section 1 implies that  $W \rightarrow W_{\cdot-t} + (A_{-t} - Y_{-t}) \mathbb{1} = W^{-t}$  is an injection. Let us also demonstrate that

$$Q_\nu(\{W_{\cdot-t} + (A_{-t} - Y_{-t}) \mathbb{1} : W \in \Omega\}) = 1. \quad (3.132)$$

By (3.131), for  $Q_\nu$ -a.e.  $V \in \Omega$  and every  $m \in \mathbb{N}$ , there is a  $\rho(m, V) \in F$  such that with  $W(m, V) := V_{+t} + (\rho(m, V) - V_t)\mathbb{I}$ ,  $A^{(m)} \equiv A^{(m)}(W(m, V))$ , and  $Y^{(m)} \equiv Y^{(m)}(W(m, V))$  we have  $W_0(m, V) \in D^n$  and

$$V = W_{-t}(m, V) + \left(A_{-t}^{(m)} - Y_{-t}^{(m)}\right) \mathbb{I}.$$

In particular, we have  $\rho(m, V) = W_0(m, V) \in D^n$ . We also recall that by the conventions of Subsection 2.1  $D$  is bounded. Therefore we may choose an accumulation point  $\rho$  of the sequence  $\rho(m, V)$ ,  $m \in \mathbb{N}$ , and a subsequence  $\rho(m_k, V)$ ,  $k \in \mathbb{N}$ , with  $\rho(m_k, V) \xrightarrow[k \rightarrow \infty]{} \rho$ . In other words,  $W(m_k; V) \xrightarrow[k \rightarrow \infty]{} W := V_{+t} + (\rho - V_t)\mathbb{I}$ . By the hypothesis that  $A_{-t} - Y_{-t}$  jumps only on a set of  $Q_\nu$ -measure zero in condition (2) of Section 1 and by condition (4) (i) this implies  $A_{-t}^{(m_k)} - Y_{-t}^{(m_k)} \xrightarrow[k \rightarrow \infty]{} A_{-t} - Y_{-t}$ . We have shown that for  $Q_\nu$ -a.e.  $V \in \Omega$  there exist a  $W \in \Omega$ , namely  $W = V_{+t} + (\rho - V_t)\mathbb{I} = V_{+t} + \lim_{k \rightarrow \infty} (\rho(m_k, V) - V_t)\mathbb{I}$ , such that

$$V = W_{-t} + (A_{-t} - Y_{-t}) \mathbb{I}$$

which is an equivalent formulation of (3.132).

As a further preparation for the next step we mention that because of the injectivity of  $W \rightarrow W_{-t} + (A_{-t} - Y_{-t}) \mathbb{I}$  for  $Q_\nu$ -a.e.  $V \in \Omega$  there is just one accumulation point of the sequence  $\rho(m, V)$ ,  $m \in \mathbb{N}$ , i. e., the limit  $\rho = \lim_{m \rightarrow \infty} \rho(m, V)$  is well-defined.

*Step 5* We shall verify conditions (i) and (iii) of Theorem 2.1 and establish the limit

$$\lim_{n \rightarrow \infty} \frac{dQ_\nu \circ f_n^{-1}}{dQ_\nu} \equiv \lim_{m \rightarrow \infty} \frac{dQ_\nu \circ (f^{(m)})^{-1}}{dQ_\nu}$$

in  $L^1(\Omega, Q_\nu)$ . In the context of Theorem 2.1 let  $M := \Omega$  and  $\mu := Q_\nu$ . By the result of Step 4 we can introduce  $f_n \equiv f^{(m)}$  and  $f$  for  $Q_\nu$ -a.e.  $W \in \Omega$  implicitly by their inverse functions

$$\begin{aligned} f^{-1}(W) &:= W_{-t} + (A_{-t} - Y_{-t}) \mathbb{I}, \\ f_n^{-1}(W) &\equiv (f^{(m)})^{-1}(W) := W_{-t} + \left(A_{-t}^{(m)} - Y_{-t}^{(m)}\right) \mathbb{I}, \end{aligned} \tag{3.133}$$

where we stress that from here on  $n \in \mathbb{N}$  has a different meaning than in Steps 1-3. Using the notation of Step 4 we have for  $Q_\nu$ -a.e.  $V \in \Omega$

$$f(V) = V_{+t} + (\rho - V_t)\mathbb{I},$$

$$f_n(V) \equiv f^{(m)}(V) = V_{+t} + (\rho(m, V) - V_t)\mathbb{I}.$$

and

$$f_n(V) \xrightarrow[n \rightarrow \infty]{} f(V),$$

i. e., we have condition (iii) of Theorem 2.1.

Let  $W \in \Omega$  with  $W_s = y + \sum_{i=1}^\infty x_i \cdot \int_0^s H_i(u) du$ ,  $s \in \mathbb{R}$ . From (3.133) we obtain

$$\begin{aligned} f_n^{-1}(W) &= W_{-t} + \left(A_{-t}^{(m)} - Y_{-t}^{(m)}\right) \mathbb{I} \\ &= (W_{-t})^{(m)} + \left(A_{-t}^{(m)} - Y_{-t}^{(m)}\right) \mathbb{I} + (W_{-t} - (W_{-t})^{(m)}) \mathbb{I}. \end{aligned}$$

From this and the fact that  $A_{-t}^{(m)} = A_{-t}(W^{(m)})$  as well as  $Y_{-t}^{(m)} = Y_{-t}(W^{(m)})$  are by definition independent of  $\xi_i$ ,  $i \notin J(m)$ , we conclude that  $Q_\nu \circ f_n^{-1}$  factorizes into

$$\begin{aligned} dQ_\nu \circ f_n^{-1}(W) &\equiv Q_\nu(df_n^{-1}(W)) \\ &= Q_\nu^{(m)}\left(d\left((W_{-t})^{(m)} + \left(A_{-t}^{(m)} - Y_{-t}^{(m)}\right)\mathbb{1}\right)\right) \times Q_\nu(\xi_i \in dx_{i'} : i \notin J(m)) . \end{aligned} \quad (3.134)$$

Let us recall that  $(W_{-t})^{(m)} = W_{-t}^{(m)}$  and therefore  $dQ_\nu^{(m)} \circ f_n^{-1}(W) = Q_\nu^{(m)}(df_n^{-1}(W)) = Q_\nu^{(m)}\left(d\left((W_{-t})^{(m)} + \left(A_{-t}^{(m)} - Y_{-t}^{(m)}\right)\mathbb{1}\right)\right)$ . For  $W \in \Omega$  we obtain by using (3.134)

$$\begin{aligned} \omega_{-t}^{(m)}(W^{(m)}) &= \frac{Q_\nu^{(m)}\left(d\left((W_{-t})^{(m)} + \left(A_{-t}^{(m)} - Y_{-t}^{(m)}\right)\mathbb{1}\right)\right)}{Q_\nu^{(m)}(dW^{(m)})} \\ &= \frac{Q_\nu^{(m)}\left(d\left((W_{-t})^{(m)} + \left(A_{-t}^{(m)} - Y_{-t}^{(m)}\right)\mathbb{1}\right)\right) \cdot Q_\nu(\xi_i \in dx_{i'} : i \notin J(m))}{Q_\nu^{(m)}(dW^{(m)}) \cdot Q_\nu(\xi_i \in dx_i : i \notin J(m))} \\ &= \frac{Q_\nu\left(d\left(W_{-t} + \left(A_{-t}^{(m)} - Y_{-t}^{(m)}\right)\mathbb{1}\right)\right)}{Q_\nu(dW)} \\ &= \frac{dQ_\nu \circ f_n^{-1}}{dQ_\nu}(W) . \end{aligned} \quad (3.135)$$

We get immediately condition (i) of Theorem 2.1. Next recall conditions (1) (iv) and (4) (i) of Section 1. Furthermore, take into consideration that  $X_{-t}$ ,  $A_{-t}$ ,  $Y_{-t}$  jump only on a set of  $Q_\nu$ -measure zero, cf. conditions (2) and (3) of Section 1, that the density  $m$  is bounded and continuous, and that  $\lambda_F(D^n) < \infty$  by the boundedness of  $D$ . Using the result of Step 3, namely (3.128), relation (3.135) shows now that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{dQ_\nu \circ f_n^{-1}}{dQ_\nu} &= \lim_{m \rightarrow \infty} \left( \frac{m(X_{-t} - Y_{-t})}{m(W_0)} \cdot \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} A_{-t} - \nabla_{d, W_0} Y_{-t} \right|_i \right) \circ \pi_m \\ &= \frac{m(X_{-t} - Y_{-t})}{m(W_0)} \cdot \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} A_{-t} - \nabla_{d, W_0} Y_{-t} \right|_i \end{aligned} \quad (3.136)$$

in  $L^1(\Omega, Q_\nu)$  where  $m \rightarrow \infty$  in the second line refers to the projection  $\pi_m$ , not to the density  $m$ .

After having verified condition (ii) of Theorem 2.1, relation (3.136) will yield (1.1).

*Step 6* We verify condition (ii) of Theorem 2.1. For this, we have to demonstrate that the sequence of densities  $d\mu \circ f_n^{-1}/d\mu$ ,  $n \in \mathbb{N}$ , is uniformly integrable. By (3.128) and (3.135) this means that we have to verify uniform integrability with respect to  $Q_\nu$  of the terms

$$\begin{aligned} \frac{dQ_\nu \circ f_n^{-1}}{dQ_\nu} &= \left( \frac{m(X_{-t} - Y_{-t})}{m(W_0)} \cdot \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} A_{-t} - \nabla_{d, W_0} Y_{-t} \right|_i \right) \circ \pi_m \\ &\leq \frac{\|m\|}{m(W_0)} \cdot \sup_m \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} A_{-t} - \nabla_{d, W_0} Y_{-t} \right|_i \circ \pi_m \end{aligned}$$

where the right-hand side is in  $L^1(\Omega, Q_\nu)$  by condition (1) (iv) of Section 1. We have thus verified the uniform integrability with respect to  $Q_\nu$  of the sequence  $dQ_\nu \circ f_n^{-1}/dQ_\nu$ ,  $n \in \mathbb{N}$ .  $\square$

## 4 Partial Integration

Let us assume (1)-(5) of Section 1. Then for  $t \in \mathbb{R}$  we have according to Theorem 1.11 and Corollary 1.13

$$\begin{aligned} \rho_{-t}(X) &= \frac{P_\mu(dX_{\cdot-t})}{P_\mu(dX)} \\ &= \frac{m(X_{-t} - Y_{-t} \circ u^{-1})}{m(u_0^{-1})} \cdot \prod_{i=1}^{n \cdot d} \left| e + \nabla_{d, W_0} A_{-t} \circ u^{-1} - \nabla_{d, W_0} Y_{-t} \circ u^{-1} \right|_i. \end{aligned}$$

In contrast to the previous sections, where  $\mu$  was used as a part of the symbol  $P_\mu = Q_\nu \circ u^{-1}$ , we give  $\mu$  now a meaning by  $\mu(A) := P_\mu(X_0 \in A)$ ,  $A \in \mathcal{B}(F)$ . For  $x \in D^n$ , let  $E_x$  denote the expectation relative to  $P_x := P_\mu(\cdot | X_0 = x)$  and let  $E_\mu$  denote the expectation relative to  $P_\mu$ . In addition to condition (4) (i) of Section 1 we may formulate the following condition.

- (6) For all  $W \in \Omega$ , the jump times for  $\nabla_{d, W_0} A(W)$  and  $\nabla_{d, W_0} Y(W)$  are a subset of  $\{\tau_k(u(W)) \equiv \tau_k(X) : k \in \mathbb{Z} \setminus \{0\}\}$ .

We may also assume that

(7)

$$\frac{d\mu}{d\nu} \in C_b(D^n) \quad \text{and} \quad \frac{d\mu}{d\nu} > 0.$$

We recall that, by condition (2) (i) of Section 1, for any fixed  $s \in \mathbb{R}$  and  $\lambda_F$ -a.e.  $x \in D^n$ ,  $A_s^2 \circ u^{-1}$  jumps only on a  $P_x$ -zero set. As a consequence of this and conditions (6) and (7) as well as conditions (1) (iv) and (4) (i) of Section 1 we obtain

$$\lim_{t \downarrow 0} \frac{P_\mu(X_t \in dx)}{P_\mu(X_0 \in dx)} = \lim_{t \downarrow 0} \int \rho_{-t}(X) P_x(dX) = \mathbb{I} \quad (4.1)$$

boundedly for  $\lambda_F$ -a.e.  $x \in D^n$ . Let us set

$$\begin{aligned} r_{-t}(X, x) &:= E_x \left( \frac{m(W_0^{-t} \circ u^{-1})}{m(u_0^{-1})} \prod_{i=1}^{n \cdot d} \left| e + \nabla_{d, W_0} A_{-t} \circ u^{-1} - \nabla_{d, W_0} Y_{-t} \circ u^{-1} \right|_i \middle| X_{-t} \right) \\ &= E_\mu \left( \frac{P_\mu(dX_{\cdot-t})}{P_\mu(dX)} \middle| X_{-t}, X_0 = x \right), \quad x \in D^n, t \in \mathbb{R}. \end{aligned} \quad (4.2)$$

Moreover, for  $x \in D^n$ , let  $E_{t,x}$  be the expectation relative to  $P_\mu(\cdot | X_t = x)$ . Let  $D(A)$  and  $D(A^*)$  denote the set of  $f \in L^2(F, \mu)$  for which, respectively, the limit

$$Af := \lim_{t \downarrow 0} \frac{1}{t} (E_{t,x} f(X_t) - f) \quad \text{and} \quad A^* f := - \lim_{t \downarrow 0} \frac{1}{t} (E_{t,x} f(X_0) - f)$$

exists in  $L^2(F, \boldsymbol{\mu})$ .

Let us write  $\langle \cdot, \cdot \rangle_\mu$  and  $\| \cdot \|_\mu$  for the inner product and the norm in  $L^2(F, \boldsymbol{\mu})$ . For  $F = \mathbb{R}^{n \cdot d}$  and  $f \equiv (f_1, \dots, f_{n \cdot d}) \in C^1(\overline{D^n}; F)$  let  $\nabla_{d,x} f := \left( \frac{\partial f_1}{\partial x_1}, \dots, \frac{\partial f_{n \cdot d}}{\partial x_{n \cdot d}} \right)$ .

Furthermore, let  $C^{1,1}(\mathbb{R} \times \overline{D^n}; F)$  denote the space of all functions  $a \equiv a(s, x)$ ,  $s \in \mathbb{R}$ ,  $x \in \overline{D^n}$ , for which  $a, a' \equiv (\partial a)/(\partial s), \nabla_{d,x} a, (\nabla_{d,x} a)' \in C(\mathbb{R} \times \overline{D^n}; F)$ . It follows that  $\nabla_{d,x}(a') \in C(\mathbb{R} \times \overline{D^n}; F)$  and that  $\nabla_{d,x}(a') = (\nabla_{d,x} a)'$ .

In the remainder of this section we will use the letter  $Z$  to abbreviate  $A - Z$ . This also means that, for example, we write  $\nabla_{d,W_0} Z_{-t} \circ u^{-1}$  for  $\nabla_{d,W_0} A_{-t} \circ u^{-1} - \nabla_{d,W_0} Y_{-t} \circ u^{-1}$  and  $\nabla_{d,W_0} Z'_0 \circ u^{-1}$  for  $\nabla_{d,W_0} A'_0 \circ u^{-1} - \nabla_{d,W_0} Y'_0 \circ u^{-1}$ .

Now we are able to state and prove the  $L^2$ -version of the integration by parts theorem.

**Theorem 4.1** *Suppose that the hypotheses of Theorem 1.11 and Corollary 1.13 as well as conditions (6) and (7) of this section are satisfied. In addition, assume the following.*

(i) *The derivative*

$$r^* := \lim_{t \downarrow 0} \frac{1}{t} (E.r_{-t}(X, \cdot) - \mathbb{I})$$

*exists in  $L^2(F, \boldsymbol{\mu})$ .*

(a) *Then for  $f \in D(A)$  and  $g \in D(A^*) \cap B_b(D^n)$ , we have*

$$\langle Af, g \rangle_\mu + \langle f, A^*g \rangle_\mu = \langle f, r^* \cdot g \rangle_\mu.$$

(b) *Let, in addition to (i),*

(ii) *the limit*

$$B^*m := -\lim_{t \downarrow 0} \frac{1}{t} (E.m(W_0^{-t} \circ u^{-1}) - m)$$

*exists in  $L^2(D^n, \lambda_F)$  and*

(iii)  *$A \equiv A_s(W - W_0, W_0) \in C^{1,1}(\mathbb{R} \times \overline{D^n}; F)$   $Q_x$ -a.e. and, for the restriction of  $\nabla_{d,W_0} A$  to  $[-1, 0]$ ,  $\nabla_{d,W_0} A' \in L^\infty(\Omega, Q_x; C_b([-1, 0]; F))$  for a.e.  $x \in D^n$ , and the same for  $Y$  then*

$$r^* = -\frac{B^*m}{m} - E. \langle \mathbf{e}, \nabla_{d,W_0} Z'_0 \circ u^{-1} \rangle_F.$$

Proof. (a) By (4.1) and condition (i) we have for  $g \in D(A^*) \cap B_b(D^n)$

$$\begin{aligned} & \lim_{t \downarrow 0} \frac{1}{t} \left( \int g(X_{-t}) \cdot r_{-t}(X, \cdot) dP - g \right) \\ &= \lim_{t \downarrow 0} \frac{1}{t} \int (g(X_{-t}) - g) \cdot r_{-t}(X, \cdot) dP + \lim_{t \downarrow 0} \frac{1}{t} \int g \cdot (r_{-t}(X, \cdot) - \mathbb{I}) dP \\ &= \lim_{t \downarrow 0} \frac{1}{t} \int (g(X_{-t}) - g) \cdot E_\mu \left( \frac{P_\mu(dX_{-t})}{P_\mu(dX)} \middle| X_{-t}, X_0 = \cdot \right) dP + r^* \cdot g \\ &= \lim_{t \downarrow 0} \frac{1}{t} \int E_\mu \left( (g(X_{-t}) - g) \cdot \frac{P_\mu(dX_{-t})}{P_\mu(dX)} \middle| X_{-t}, X_0 = \cdot \right) dP + r^* \cdot g \\ &= \lim_{t \downarrow 0} \left( \frac{1}{t} \int (g(X_0) - g) P_\mu(dX|X_t = \cdot) \frac{P_\mu(X_t \in d \cdot)}{P_\mu(X_0 \in d \cdot)} \right) + r^* \cdot g \\ &= -A^*g + r^* \cdot g \quad \text{in } L^2(F, \boldsymbol{\mu}). \end{aligned} \tag{4.3}$$

Keeping (4.2) and (4.3) in mind, the following chain of equations is now self-explaining. It holds for  $f \in D(A)$  and  $g \in D(A^*) \cap B_b(D^n)$  that

$$\begin{aligned}
\langle Af, g \rangle_\mu &= \left\langle \frac{d^+}{dt} \Big|_{t=0} E.f(X_t), g \right\rangle_\mu \\
&= \frac{d^+}{dt} \Big|_{t=0} \langle E.f(X_t), g \rangle_\mu \\
&= \lim_{t \downarrow 0} \frac{1}{t} \left( \int f(X_t) g(X_0) dP_\mu - \langle f, g \rangle_\mu \right) \\
&= \lim_{t \downarrow 0} \frac{1}{t} \left( \int f(X_0) g(X_{-t}) P_\mu(dX_{-t}) - \langle f, g \rangle_\mu \right) \\
&= \lim_{t \downarrow 0} \frac{1}{t} \left( \int f(X_0) g(X_{-t}) \cdot E_\mu \left( \frac{P_\mu(dX_{-t})}{P_\mu(dX)} \Big| X_{-t}, X_0 \right) dP_\mu - \langle f, g \rangle_\mu \right) \\
&= \lim_{t \downarrow 0} \frac{1}{t} \int f(X_0) \left( g(X_{-t}) \cdot E_\mu \left( \frac{P_\mu(dX_{-t})}{P_\mu(dX)} \Big| X_{-t}, X_0 \right) - g(X_0) \right) dP_\mu \\
&= \lim_{t \downarrow 0} \frac{1}{t} \int f(x) \left( \int g(X_{-t}) \cdot E_\mu \left( \frac{P_\mu(dX_{-t})}{P_\mu(dX)} \Big| X_{-t}, X_0 = x \right) dP_x - g(x) \right) \mu(dx) \\
&= \lim_{t \downarrow 0} \frac{1}{t} \int f(x) \left( \int g(X_{-t}) \cdot r_{-t}(X, x) dP_x - g(x) \right) \mu(dx) \\
&= -\langle f, A^*g \rangle_\mu + \langle f, r^* \cdot g \rangle_\mu. \tag{4.4}
\end{aligned}$$

We obtain part (a).

(b) By (4.2) and (ii) and (iii) of the present theorem there is a sequence  $t_n > 0$ ,  $n \in \mathbb{N}$ , with  $t_n \xrightarrow{n \rightarrow \infty} 0$  such that  $\mu$ -a.e.

$$\begin{aligned}
&\left| E. \left( \frac{r_{-t_n}(X, \cdot) - \mathbb{I}}{t_n} \right) + \frac{B^*m}{m} + E. \langle \mathbf{e}, \nabla_{d, W_0} Z'_0 \circ u^{-1} \rangle_F \right| \\
&= \left| \frac{1}{t_n} \left( E. \left( \frac{m(W_0^{-t_n} \circ u^{-1})}{m(u_0^{-1})} \cdot \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} Z_{-t_n} \circ u^{-1} \right|_i \right) - \mathbb{I} \right) \right. \\
&\quad \left. + \frac{B^*m}{m} + E. \langle \mathbf{e}, \nabla_{d, W_0} Z'_0 \circ u^{-1} \rangle_F \right| \\
&\leq \left| \frac{1}{t_n} E. \left( \frac{m(W_0^{-t_n} \circ u^{-1})}{m(u_0^{-1})} \cdot \left( \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} Z_{-t_n} \circ u^{-1} \right|_i - \mathbb{I} \right) \right) \right. \\
&\quad \left. + E. \langle \mathbf{e}, \nabla_{d, W_0} Z'_0 \circ u^{-1} \rangle_F \right| + \left| \frac{1}{t_n} E. \left( \frac{m(W_0^{-t_n} \circ u^{-1})}{m(u_0^{-1})} - \mathbb{I} \right) + \frac{B^*m}{m} \right| \\
&\xrightarrow{n \rightarrow \infty} 0.
\end{aligned}$$

Part (b) is now a consequence of condition (i) of this theorem.  $\square$

Let us turn to the *weak version* of the integration by parts theorem. For this, let  $D(A_w)$  denote the set of all  $f \in C_b(D^n)$  for which the limit

$$A_w f(g) := \lim_{t \downarrow 0} \frac{1}{t} \int g(E.f(X_t) - f) d\mu$$

exists for all  $g \in C_b(D^n)$ . Furthermore, let  $D(A_w^*)$  be the set of all  $g \in C_b(D^n)$  for which the limit

$$A_w^*g(f) := -\lim_{t \downarrow 0} \frac{1}{t} \int f(E_{t, \cdot}g(X_0) - g) d\mu \quad (4.5)$$

exists for all  $f \in C_b(D^n)$  such that  $A_w^*g$  is a bounded linear functional on  $C(\overline{D^n})$ . Let us recall conditions (1) (iv) and (3) (iii) of Section 1 and introduce the following condition.

- (8) We have  $\sup |\nabla_{d, W_0} Z_{-t}(W)| < 1$  where  $|\cdot|$  refers to the maximal coordinate wise absolute value and the supremum is taken over all  $W \in \bigcup_m \{\pi_m V : V \in \Omega\}$  and  $t \in [0, 1]$ .

By conditions 2 (i) and (4) (i) of Section 1 we obtain in a coordinate wise way

$$\log(\mathbf{e} + \nabla_{d, W_0} Z_{-t}) = \log|\mathbf{e} + \nabla_{d, W_0} Z_{-t}| \quad Q_\nu\text{-a.e.}$$

and thus also  $Q_{\lambda_F}\text{-a.e.}, t \in [0, 1]$ .

**Theorem 4.2** *Suppose that the hypotheses of Theorem 1.11 and Corollary 1.13 as well as conditions (6)-(8) of this section are satisfied.*

(a) *In addition, assume the following.*

(iv) *The limit*

$$r_w^*(h) := \lim_{t \downarrow 0} \frac{1}{t} \int h(E.r_{-t}(X, \cdot) - \mathbb{I}) d\mu$$

*exists for all  $h \in C_b(D^n)$ .*

*Then for  $f \in D(A_w)$  and  $g \in D(A_w^*)$  we have*

$$A_w f(g) + A_w^* g(f) = r_w^*(fg).$$

(b) *Assume the following.*

(v) *The limit*

$$B_w^*m(h) := -\lim_{t \downarrow 0} \frac{1}{t} \int h(E.m(W_0^{-t} \circ u^{-1}) - m) dx$$

*exists for all  $h \in C_b(D^n)$ .*

(vi)

$$\limsup_{t \downarrow 0} \frac{1}{t} \int (m(W_0^{-t} \circ u^{-1}) - m(u_0^{-1}))^2 dP_{\lambda_F} < \infty.$$

(vii) *The limit*

$$e_w^*(h) := \lim_{t \downarrow 0} \frac{1}{t} \int h E. \langle \mathbf{e}, \nabla_{d, W_0} Z_{-t} \circ u^{-1} \rangle_F d\mu$$

*exists for all  $h \in C_b(D^n)$ .*

(viii) For all  $\mathbf{g} \in \mathbb{R}^{n \cdot d}$  with  $\max_{j \in \{1, \dots, n \cdot d\}} |\mathbf{g}_j| \leq 1$  and all  $h \in C_b(D^n)$  there exists  $d_h > 0$  such that for all  $k \geq 1$  and  $t \in [0, 1]$ ,

$$\left| \int h E. \left( \langle \mathbf{g}, \nabla_{d, W_0} Z_{-t} \circ u^{-1} \rangle_F \right)^k dx \right| \leq d_h t^k.$$

Then we have (iv) and

$$r_w^*(h) = -B_w^* m(h) - e_w^*(h), \quad h \in C_b(D^n).$$

Proof. (a) We modify the proof of Theorem 4.1 (a). By (4.1) and (iv) we verify the weak version of (4.3),

$$\begin{aligned} & \lim_{t \downarrow 0} \frac{1}{t} \int f \left( \int g(X_{-t}) \cdot r_{-t}(X, \cdot) dP - g \right) d\mu \\ &= \lim_{t \downarrow 0} \frac{1}{t} \int f \left( \int (g(X_{-t}) - g) \cdot r_{-t}(X, \cdot) dP \right) d\mu \\ & \quad + \lim_{t \downarrow 0} \frac{1}{t} \int f \int g \cdot (r_{-t}(X, \cdot) - \mathbb{1}) dP d\mu \\ &= \lim_{t \downarrow 0} \frac{1}{t} \int f(x) (E_{t,x} g(X_0) - g(x)) \frac{P_\mu(X_t \in dx)}{P_\mu(X_0 \in dx)} \mu(dx) + r_w^*(fg) \\ &= r_w^*(fg) - A_w^* g(f), \end{aligned}$$

for  $f \in D(A_w)$  and  $g \in D(A_w^*)$ . Anything else is now as in (4.4).

(b) Let  $h \in C_b(D^n)$ . By the above condition (8) and assumptions (vii) and (viii) it turns out that

$$\frac{1}{t} \int h E. \left( \prod_{i=1}^{n \cdot d} |\mathbf{e} + \nabla_{d, W_0} Z_{-t} \circ u^{-1}| - \mathbb{1} \right) d\mu \xrightarrow{t \downarrow 0} -e_w^*(h). \quad (4.6)$$

Recalling condition (8) and that, by the above condition (7) it holds that  $d\mu/d\nu \in C_b(D^n)$ , we obtain by (vi) and (viii)

$$\begin{aligned} & \left( \frac{1}{t} \int h(X_0) \left( \frac{m(W_0^{-t} \circ u^{-1})}{m(u_0^{-1})} - \mathbb{1} \right) \cdot \left( \prod_{i=1}^{n \cdot d} |\mathbf{e} + \nabla_{d, W_0} Z_{-t} \circ u^{-1}| - \mathbb{1} \right) dP_\mu \right)^2 \\ & \leq \frac{1}{t} \int (m(W_0^{-t} \circ u^{-1}) - m(u_0^{-1}))^2 dP_{\lambda_F} \times \\ & \quad \times \frac{1}{t} \int h(X_0)^2 \frac{d\mu}{d\nu}(X_0)^2 \left( \prod_{i=1}^{n \cdot d} |\mathbf{e} + \nabla_{d, W_0} Z_{-t} \circ u^{-1}| - \mathbb{1} \right)^2 dP_{\lambda_F} \\ & \xrightarrow{t \downarrow 0} 0. \end{aligned} \quad (4.7)$$

It follows now from (4.6) and (4.7) that

$$\lim_{t \downarrow 0} \frac{1}{t} \int h E. \left( \frac{m(W_0^{-t} \circ u^{-1})}{m} \cdot \left( \prod_{i=1}^{n \cdot d} |\mathbf{e} + \nabla_{d, W_0} Z_{-t} \circ u^{-1}| - \mathbb{1} \right) \right) d\mu$$

$$\begin{aligned}
&= \lim_{t \downarrow 0} \frac{1}{t} \int h E. \left( \prod_{i=1}^{n-d} \left| e + \nabla_{d, W_0} Z_{-t} \circ u^{-1} \right| - \mathbb{I} \right) d\mu \\
&\quad + \lim_{t \downarrow 0} \frac{1}{t} \int h(X_0) \left( \frac{m(W_0^{-t} \circ u^{-1})}{m(u_0^{-1})} - \mathbb{I} \right) \cdot \left( \prod_{i=1}^{n-d} \left| e + \nabla_{d, W_0} Z_{-t} \circ u^{-1} \right| - \mathbb{I} \right) dP_\mu \\
&= -e_w^*(h). \tag{4.8}
\end{aligned}$$

Recalling (4.2) and condition (v) we obtain from (4.8)

$$\begin{aligned}
-B_w^* m(h) - e_w^*(h) &= \lim_{t \downarrow 0} \frac{1}{t} \int h E. \left( \frac{m(W_0^{-t} \circ u^{-1})}{m} - \mathbb{I} \right) d\mu \\
&\quad + \lim_{t \downarrow 0} \frac{1}{t} \int h E. \left( \frac{m(W_0^{-t} \circ u^{-1})}{m} \cdot \left( \prod_{i=1}^{n-d} \left| e + \nabla_{d, W_0} Z_{-t} \circ u^{-1} \right| - \mathbb{I} \right) \right) d\mu \\
&= \lim_{t \downarrow 0} \frac{1}{t} \int h (E.r_{-t}(X, \cdot) - \mathbb{I}) d\mu \\
&= r_w^*(h), \quad h \in C_b(D^n).
\end{aligned}$$

□

Let  $\mathcal{C}$  be the set of all  $f \in C_b(D^n)$  such that for all  $g \in D(A_w)$  the limit

$$R(f, g) := \lim_{t \downarrow 0} \frac{1}{t} \int (f(X_t) - f(X_0)) (g(X_t) - g(X_0)) dP_\mu$$

exists. Denote by  $D(A_w^*, \mathcal{C})$  the set of all  $g \in C_b(D^n)$  for which the limit (4.5) exists for all  $f \in \mathcal{C}$ .

**Proposition 4.3** *Suppose that the hypotheses of Theorem 1.11 and Corollary 1.13 as well as conditions (6)-(8) of this section are satisfied.*

(a) *Furthermore assume (v) and (viii) of Theorem 4.2 and the following.*

(ix)  *$h \rightarrow B_w^* m(h)$  is a bounded linear functional on  $C(\overline{D^n})$ .*

*It holds that  $D(A_w) \subseteq D(A_w^*, \mathcal{C})$ . For  $f \in \mathcal{C}$  and  $g \in D(A_w)$  we have*

$$A_w^* g(f) = A_w g(f) + R(f, g).$$

(b) *Assume in addition that*

(x) *the process  $Y$  is constant zero or, equivalently,  $X_0 = W_0$   
and  $m \in D(A_w)$ .*

*Then for all  $h \in C_b(D^n)$  such that  $h/m \in \mathcal{C}$  we have  $m \in D(A_w^*, \mathcal{C})$  and*

$$B_w^* m(h) = A_w^* m(h/m) = A_w m(h/m) + R(h/m, m).$$

Proof. (a) By (6) and (7) as well as its consequences outlined in the beginning of this section, we have

$$\begin{aligned}
& \frac{P_\mu(X_t \in dx)}{P_\mu(X_0 \in dx)} E_{t,x}(g(X_0) - g(x)) = \frac{P_\mu(X_t \in dx)}{P_\mu(X_0 \in dx)} E_{t,x}(g(X_0) - g(X_t)) \\
& = E_x(g(X_{-t}) - g(X_0)) \\
& \quad + \int (g(X_{-t}) - g(X_0))(\rho_{-t}(X) - \mathbb{1}) P_x(dX) \\
& = E_x(g(X_{-t}) - g(X_0)) + \frac{1}{m(x)} \int (g(X_{-t}) - g(X_0)) (m(W_0^{-t} \circ u^{-1}) \times \\
& \quad \times \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} Z_{-t} \circ u^{-1} \right|_i - m(x)) P_x(dX) \\
& \xrightarrow{t \downarrow 0} 0
\end{aligned} \tag{4.9}$$

for  $g \in C_b(D^n)$  and  $\lambda_F$ -a.e.  $x \in D^n$ . Let  $f, g \in C_b(D^n)$ . We have also

$$\begin{aligned}
& \frac{1}{t} \int f(E_{t,\cdot} g(X_0) - g) (E \cdot \rho_{-t} - \mathbb{1}) d\mu \\
& = \frac{1}{t} \int f(E_{t,\cdot} g(X_0) - g) \cdot E \cdot \left( \frac{m(W_0^{-t} \circ u^{-1})}{m} \cdot \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} Z_{-t} \circ u^{-1} \right|_i - \mathbb{1} \right) d\mu \\
& = \int f(E_{t,\cdot} g(X_0) - g) \cdot E \cdot \left( \frac{m(W_0^{-t} \circ u^{-1}) - m}{t} \right) dx \\
& \quad + \int f(E_{t,\cdot} g(X_0) - g) \times \\
& \quad \times E \cdot \left( \frac{m(W_0^{-t} \circ u^{-1}) - m}{t} \cdot \left( \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} Z_{-t} \circ u^{-1} \right|_i - \mathbb{1} \right) \right) dx \\
& \quad + \int f(E_{t,\cdot} g(X_0) - g) \cdot \frac{1}{t} \left( E \cdot \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} Z_{-t} \circ u^{-1} \right|_i - \mathbb{1} \right) d\mu \\
& = T_1(t) + T_2(t) + T_3(t).
\end{aligned} \tag{4.10}$$

Now we take into consideration that because of (4.9) we have  $\lim_{t \downarrow 0} (E_{t,\cdot} g(X_0) - g) = 0$   $\lambda_F$ -a.e. on  $D^n$ . With this,  $T_1(t)$  converges to zero as  $t \downarrow 0$  by (ix) of the present proposition. Let us focus on  $T_2(t)$ . Taking into consideration condition (8) of this section, we get

$$\begin{aligned}
(T_2(t))^2 & \leq \int f^2(E_{t,\cdot} g(X_0) - g)^2 dx \times \\
& \quad \times \int \left( E \cdot \left( \frac{m(W_0^{-t} \circ u^{-1}) - m}{t} \left( \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} Z_{-t} \circ u^{-1} \right|_i - \mathbb{1} \right) \right) \right)^2 dx \\
& \leq \int f^2(E_{t,\cdot} g(X_0) - g)^2 dx \times
\end{aligned}$$

$$\begin{aligned}
& \times \int E. \left( \frac{m(W_0^{-t} \circ u^{-1}) - m}{t} \right)^2 \cdot E. \left( \prod_{i=1}^{n \cdot d} \left| e + \nabla_{d, W_0} Z_{-t} \circ u^{-1} \right|_i - \mathbb{1} \right)^2 dx \\
& \leq 4\|m\|^2 \cdot \left( \int f^2 (E_{t, \cdot} g(X_0) - g)^2 dx \right)^2 \times \\
& \quad \times \frac{1}{t^2} \int E. \left( \prod_{i=1}^{n \cdot d} \left| e + \nabla_{d, W_0} Z_{-t} \circ u^{-1} \right| - \mathbb{1} \right)^2 dx \xrightarrow{t \downarrow 0} 0
\end{aligned}$$

the limit in the last line is a consequence of (4.1), (4.9) and condition (viii) of Theorem 4.2. For  $T_3(t)$ , we recall condition (7) of this section. Relations (4.1), (4.9) and assumption (viii) of Theorem 4.2 show also here that

$$\begin{aligned}
(T_3(t))^2 & \leq \int f^2 (E_{t, \cdot} g(X_0) - g)^2 d\boldsymbol{\mu} \times \\
& \quad \times \frac{1}{t^2} \int E. \left( \prod_{i=1}^{n \cdot d} \left| e + \nabla_{d, W_0} Z_{-t} \circ u^{-1} \right| - \mathbb{1} \right)^2 d\boldsymbol{\mu} \xrightarrow{t \downarrow 0} 0.
\end{aligned}$$

From (4.10) we get now

$$\lim_{t \downarrow 0} \frac{1}{t} \int f (E_{t, \cdot} g(X_0) - g) (E. \rho_{-t} - \mathbb{1}) d\boldsymbol{\mu} = 0, \quad f, g \in C_b(D^n).$$

Furthermore, it holds that

$$P_\mu(X_t \in dx) = E_\mu(\rho_{-t} | X_0 = x) \cdot \boldsymbol{\mu}(dx) = E_{x\rho_{-t}} \cdot \boldsymbol{\mu}(dx).$$

We get with this

$$\begin{aligned}
& - \lim_{t \downarrow 0} \frac{1}{t} \int f (E_{t, \cdot} g(X_0) - g) d\boldsymbol{\mu} \\
& = - \lim_{t \downarrow 0} \frac{1}{t} \int f (E_{t, \cdot} g(X_0) - g) E. \rho_{-t} d\boldsymbol{\mu} \\
& \quad + \lim_{t \downarrow 0} \frac{1}{t} \int f (E_{t, \cdot} g(X_0) - g) (E. \rho_{-t} - \mathbb{1}) d\boldsymbol{\mu} \\
& = - \lim_{t \downarrow 0} \frac{1}{t} \int f E_{t, \cdot} (g(X_0) - g) dP_\mu(X_t \in \cdot) \\
& = - \lim_{t \downarrow 0} \frac{1}{t} \int f(X_t) (g(X_0) - g(X_t)) dP_\mu \\
& = \lim_{t \downarrow 0} \frac{1}{t} \int f(X_0) (g(X_t) - g(X_0)) dP_\mu \\
& \quad + \lim_{t \downarrow 0} \frac{1}{t} \int (f(X_t) - f(X_0)) (g(X_t) - g(X_0)) dP_\mu \\
& = A_w g(f) + R(f, g), \quad f \in \mathcal{C}, \quad g \in D(A_w).
\end{aligned} \tag{4.11}$$

We obtain  $g \in D(A_w^*, \mathcal{C})$  as well as  $A_w^* g(f) = A_w g(f) + R(f, g)$ .

(b) By (x) and the definition of  $D(A_w)$  we have  $m \in C_b^1(D^n)$  and by (v) it holds that

$$\begin{aligned}
B_w^* m(h) &= -\lim_{t \downarrow 0} \frac{1}{t} \int \frac{h}{m} (E.m(X_{-t}) - m) \boldsymbol{\mu}(dx) \\
&= -\lim_{t \downarrow 0} \frac{1}{t} \int \frac{h(X_0)}{m(X_0)} (m(X_{-t}) - m(X_0)) dP_\mu \\
&= -\lim_{t \downarrow 0} \frac{1}{t} \int \frac{h(X_0)}{m(X_0)} (m(X_{-t}) - m(X_0)) \rho_{-t}(X) dP_\mu \\
&\quad - \lim_{t \downarrow 0} \frac{1}{t} \int \frac{h(X_0)}{m(X_0)} (m(X_{-t}) - m(X_0)) (\mathbb{I} - \rho_{-t}(X)) dP_\mu \\
&= -\lim_{t \downarrow 0} \frac{1}{t} \int \frac{h(X_t)}{m(X_t)} (m(X_0) - m(X_t)) dP_\mu \\
&\quad - \lim_{t \downarrow 0} \frac{1}{t} \int \frac{h(X_0)}{m(X_0)} (m(X_{-t}) - m(X_0)) (\mathbb{I} - \rho_{-t}(X)) dP_\mu \tag{4.12}
\end{aligned}$$

for all  $h \in C_b(D^n)$  such that  $h/m \in C_b(D^n)$ . The last line of (4.12) is zero, by

$$\begin{aligned}
&\left( \frac{1}{t} \int \frac{h(X_0)}{m(X_0)} (m(X_{-t}) - m(X_0)) (\mathbb{I} - \rho_{-t}(X)) dP_\mu \right)^2 \\
&= \left( \frac{1}{t} \int \frac{h(X_0)}{m(X_0)} (m(X_{-t}) - m(X_0)) (m(X_0) - m(X_0) \rho_{-t}(X)) dP_{\lambda_F} \right)^2 \\
&\leq \left\| \frac{h}{m} \right\|^2 \frac{1}{t} \int ((m(X_{-t}) - m(X_0))^2 dP_{\lambda_F} \cdot \frac{1}{t} \int (m(X_0) - m(X_0) \rho_{-t}(X))^2 dP_{\lambda_F} \\
&\leq \left\| \frac{h}{m} \right\|^2 \frac{1}{t} \int ((m(X_{-t}) - m(X_0))^2 dP_{\lambda_F} \times \\
&\quad \times \frac{1}{t} \int \left( m(X_0) - m(X_{-t}) \cdot \prod_{i=1}^{n-d} \left| e + \nabla_{d, W_0} Z_{-t} \circ u^{-1} \right|_i \right)^2 dP_{\lambda_F} \xrightarrow{t \downarrow 0} 0
\end{aligned}$$

as well as condition (8) of this section and hypotheses (vi) and (viii) of Theorem 4.2. Comparing the second last line of (4.12) with (4.11) it turns out that

$$B_w^* m(h) = A_w m(h/m) + R(h/m, m) = A_w^* m(h/m)$$

if  $m \in D(A_w)$  and  $h/m \in \mathcal{C}$  which implies  $m \in D(A_w^*, \mathcal{C})$ . □

## 5 Relative Compactness of Particle Systems

Let us assume that for all  $n \in \mathbb{N}$  we are given a system consisting of  $n$  particles  $\{X_1^n, \dots, X_n^n\}$ , each of which being a stochastic process with time domain  $\mathbb{R}$  taking values in  $D$ . We consider the measure valued stochastic process  $\mathbf{X}$  given by  $\mathbf{X}_t^n := \frac{1}{n} \sum_{i=1}^n \delta_{(X_i^n)_t}$ ,  $t \in \mathbb{R}$ .

In addition, we introduce the  $n \cdot d$ -dimensional process  $X^n \equiv (X_t^n)_{t \in \mathbb{R}} = ((X_1^n)_t, \dots, (X_n^n)_t)_{t \in \mathbb{R}}$  which we assume to be of the form  $X^n = W^n + A^n$  satisfying, for every  $n \in \mathbb{N}$ , the hypotheses of Section 1. Furthermore, we suppose that the distribution of  $X^n$  is invariant under the permutation of the  $n$  particles provided that the distribution of  $W_0^n =$

$((W_1)_0, \dots, (W_n)_0)$  is invariant under the permutation of the  $n$   $d$ -dimensional random variables  $\{(W_1)_0, \dots, (W_n)_0\}$ .

Let  $\mathcal{M}_1(\overline{D})$  denote the space of all probability measures on  $(\overline{D}, \mathcal{B}(\overline{D}))$ . Moreover, abbreviate for  $g \in C(\overline{D})$  and  $\mu \in \mathcal{M}_1(\overline{D})$  the integral  $\int g d\mu$  by  $(g, \mu)$  and define

$$\begin{aligned} \tilde{C}^1(\mathcal{M}_1(\overline{D})) \\ := \left\{ F(\mu) = \varphi((g_1, \mu), \dots, (g_k, \mu)), \ g_1, \dots, g_k \in C^1(\overline{D}), \ \varphi \in C_b^1(\mathbb{R}^k), \ k \in \mathbb{Z}_+ \right\}. \end{aligned}$$

It is our primary goal to formulate conditions in order to verify relative compactness of the family  $F(\mathbf{X}^n) \equiv (F(\mathbf{X}_t^n))_{t \geq 0}$ ,  $F \in \tilde{C}^1(\mathcal{M}_1(\overline{D}))$ ,  $n \in \mathbb{N}$ . To derive relative compactness of the family  $\mathbf{X}^n \equiv (\mathbf{X}_t^n)_{t \geq 0}$ ,  $n \in \mathbb{N}$ , from this is then standard.

As in Section 4, we use the symbol  $Z \equiv Z^n$  to abbreviate  $A - Y \equiv A^n - Y^n$ . Referring to the number of particles, here we also use the notation  $A^n = (A^1)^n + (A^2)^n$ . In contrast to Section 4, where we considered  $X$  the given process, we look in this section at the particle systems as at processes constructed from Brownian motion. Therefore we also refer to the measures  $Q_\nu \equiv Q_{\nu_n}^{(n)}$  and  $Q. \equiv Q.^{(n)}$  rather than to  $P_\mu$  and  $P.$ .

**Theorem 5.1** *For every  $n \in \mathbb{N}$ , let  $X^n$  be a stochastic process on the probability space  $(\Omega^{(n)}, \mathcal{F}^{(n)}, Q_{\nu_n}^{(n)})$  satisfying the hypotheses of Theorem 1.11 or, respectively, of Theorem 1.12. Let  $m_n$  be as in Corollary 1.13. In addition to condition (2) (i) of Subsection 1.2 or, respectively, condition (ii) of Theorem 1.12, suppose that for every  $n \in \mathbb{N}$  and  $Q_{\nu_n}^{(n)}$ -a.e.  $W \in \Omega^{(n)}$ ,  $(A^1)^n(W)$  is equi-continuous on  $\mathbb{R}$ . Assume the following.*

- (i) *For all  $n \in \mathbb{N}$ ,  $Q_{\nu_n}^{(n)}$ -almost never two of the particles  $\{X_1^n, \dots, X_n^n\}$  jump at the same time.*
- (ii) *There exists a sequence  $c_n > 0$ ,  $n \in \mathbb{N}$ , such that for all  $k \in \mathbb{N}$ , all  $j \in \{1, \dots, n \cdot d\}$ , all  $\delta \in [0, 1]$ , and all  $x \in D^n$ ,*

$$\text{ess sup}_{\Omega} \left( \sup_{t \in [0, \delta]} \left| \langle e_j, \nabla_{d, W_0} Z_{-t}^n \rangle_F \right|^k \right) \leq c_n \delta^k$$

*and  $\lim_{\delta \downarrow 0} \limsup_{n \in \mathbb{N}} c_n (1 + \delta)^n = 0$ . Here, the  $\text{ess sup}$  refers to the measure  $Q_{\nu_n}^{(n)}$ .*

- (iii) *Denoting by  $E^{(n)}$  the expectation with respect to  $Q^{(n)}$ , we have*

$$\lim_{\delta \downarrow 0} \limsup_{n \in \mathbb{N}} \int_{D^n} E_x^{(n)} \left( \sup_{t \in [0, \delta]} \left| m_n(X_{-t}^n - Y_{-t}^n) - m_n(x) \right| \right) dx = 0.$$

(a) *Let  $F \in \tilde{C}^1(\mathcal{M}_1(\overline{D}))$ . The family of stochastic processes  $F(\mathbf{X}^n) \equiv (F(\mathbf{X}_t^n))_{t \geq 0}$ ,  $n \in \mathbb{N}$ , is relatively compact with respect to the topology of weak convergence of probability measures over the Skorokhod space  $D_{[-\|F\|, \|F\|]}(\mathbb{R})$ .*

(b) *The family of stochastic processes  $\mathbf{X}^n \equiv (\mathbf{X}_t^n)_{t \geq 0}$ ,  $n \in \mathbb{N}$ , is relatively compact with respect to the topology of weak convergence of probability measures over the Skorokhod space  $D_{\mathcal{M}_1(\overline{D})}(\mathbb{R})$ .*

**Proof.** *Step 1* Let  $f \in C^1(\overline{D}^n)$  be defined by  $f(x_1, \dots, x_n) := F\left(\frac{1}{n} \sum_{i=1}^n \delta_{x_i}\right)$ ,  $x_1, \dots, x_n \in \overline{D}$ . In this step we introduce the objects used in Chapter 3 of [5], Theorem 8.6 and Remark 8.7.

We fix  $n \in \mathbb{N}$  and drop the index  $n$  from the notation since no ambiguity is possible. Let  $0 < \delta < 1$  and introduce

$$\gamma(\delta) \equiv \gamma_n(\delta; f) := \sup_{t \in \mathbb{R}, 0 \leq u \leq \delta} (f(X_{t+u}) - f(X_t))^2.$$

Furthermore, set  $\mathcal{A} = [-\delta, \delta]$ . For  $\alpha \in \mathcal{A}$ , let  $\Omega_\alpha$  be the collection of all  $u^{-1}(X) \in \Omega$  satisfying the following.

(j) There are sequences  $s_m \in \mathbb{R}$  and  $\alpha_m \in \mathbb{R}$ ,  $m \in \mathbb{N}$ , with  $\lim_{m \rightarrow \infty} \alpha_m = \alpha$  and

$$\lim_{m \rightarrow \infty} (f(X_{s_m + |\alpha_m|}) - f(X_{s_m}))^2 = \sup_{t \in \mathbb{R}, 0 \leq u \leq \delta} (f(X_{t+u}) - f(X_t))^2 =: A.$$

(jj)

$$\text{sign}(\alpha) \cdot \lim_{m \rightarrow \infty} (f(X_{s_m + |\alpha_m|}) - f(X_{s_m})) \geq 0.$$

(jjj) If  $\alpha \leq 0$  there is no pair of sequences  $s'_m \in \mathbb{R}$  and  $\alpha'_m \in \mathbb{R}$ ,  $m \in \mathbb{N}$ , with  $\lim_{m \rightarrow \infty} \alpha'_m = \alpha$  such that

$$\lim_{m \rightarrow \infty} (f(X_{s'_m + |\alpha'_m|}) - f(X_{s'_m})) > 0.$$

(jv)

$$\limsup_{m \rightarrow \infty} (f(X_{t_m + |\beta_m|}) - f(X_{t_m}))^2 < A$$

for all sequences  $t_m \in \mathbb{R}$  and  $\beta_m \in \mathbb{R}$ ,  $m \in \mathbb{N}$ , with  $\lim_{m \rightarrow \infty} \beta_m = \beta$  and  $|\beta| < |\alpha|$ .

Let  $\varepsilon > 0$ . Set  $t(\alpha) := |\alpha| - \varepsilon$ ,  $\alpha \in \mathcal{A}$ , let  $\tau^\alpha \equiv \tau^\alpha(\varepsilon)$  be the random time defined in Corollary 1.15, and denote by  $\mathbb{1}$  the function which is constant one. We have the hypotheses (i) and (ii) of Corollary 1.15 and obtain

$$\begin{aligned} \rho_{-\tau^\alpha(\varepsilon)}(X) &\xrightarrow{\varepsilon \downarrow 0} \frac{m(X_{-|\alpha|} - Y_{-|\alpha|} \circ u^{-1})}{m(u_0^{-1})} \cdot \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} Z_{-|\alpha|} \circ u^{-1} \right|_i \\ &=: \rho_{-|\alpha|}(X) \quad P_\mu\text{-a.e.} \end{aligned} \tag{5.1}$$

In addition, using the notation of the definition of the sets  $\Omega_\alpha$ , we get

$$\begin{aligned} E_\mu(\gamma(\delta)) &= E_\mu \left( \sup_{t \in \mathbb{R}, 0 \leq u \leq \delta} (f(X_{t+u}) - f(X_t))^2 \right) \\ &\leq E_\mu \left( \lim_{m \rightarrow \infty} (f(X_{s_m + |\alpha_m|}) - f(X_{s_m}))^2 \right) \\ &= \lim_{m \rightarrow \infty} E_\mu \left( (f(X_{s_m + |\alpha_m|}) - f(X_{s_m}))^2 \right) \\ &= \lim_{m \rightarrow \infty} \left( E_\mu((f(X_{s_m + |\alpha_m|}))^2) - (f(X_{s_m}))^2 \right) \end{aligned}$$

$$\begin{aligned}
& -2E_\mu \left( f(X_{s_m}) \cdot (f(X_{s_m+|\alpha_m|}) - f(X_{s_m})) \right) \\
& \leq \liminf_{m \rightarrow \infty} \left( |E_\mu ((f(X_{s_m+|\alpha_m|}))^2 - (f(X_{s_m}))^2)| \right. \\
& \quad \left. + 2\|f\| \cdot E_\mu |f(X_{s_m+|\alpha_m|}) - f(X_{s_m})| \right) \\
& = \liminf_{m \rightarrow \infty} |E_\mu ((f(X_{s_m+\alpha_m}))^2 - (f(X_{s_m}))^2)| \\
& \quad + 2\|f\| \cdot \lim_{m \rightarrow \infty} E_\mu (\text{sign}(\alpha) (f(X_{s_m+|\alpha_m|}) - f(X_{s_m}))) \\
& \leq \liminf_{m \rightarrow \infty} \left( |E_\mu ((f(X_{s_m}))^2 (\rho_{-\tau^\alpha} - \mathbb{1}))| \right. \\
& \quad \left. + E_\mu |(f(X_{s_m+|\alpha_m|}))^2 - (f(X_{s_m+\tau^\alpha}))^2| \right) \\
& \quad + 2\|f\| \cdot \lim_{m \rightarrow \infty} \left( E_\mu (\text{sign}(\alpha) f(X_{s_m}) (\rho_{-\tau^\alpha} - \mathbb{1})) \right. \\
& \quad \left. + 2\|f\| \cdot E_\mu |f(X_{s_m+|\alpha_m|}) - f(X_{s_m+\tau^\alpha})| \right).
\end{aligned}$$

We recall  $\tau^\alpha = |\alpha| - \varepsilon$  and (5.1). Letting  $\varepsilon \downarrow 0$  we obtain

$$E_\mu (\gamma(\delta)) \leq 3\|f\|^2 \cdot E_\mu |\rho_{-|\alpha|}(X) - \mathbb{1}| + 4\|f\| \cdot \liminf_{m \rightarrow \infty} E_\mu |f(X_{s_m+|\alpha_m|}) - f(X_{s_m+|\alpha|})|.$$

For the second item of the right-hand side, we use now  $\lim_{m \rightarrow \infty} |\alpha_m| = |\alpha|$  and the fact that  $W_t + A_t^1$  is equi-continuous on  $t \in [s_m, s_m + |\alpha| + 1]$ ,  $m \in \mathbb{N}$ , by Paul Lévy's modulus of continuity of  $W$  and by the equi-continuity of  $A^1$ . Furthermore we pay attention to the particular representation of  $f$ ,  $f(x_1, \dots, x_n) := F(\frac{1}{n} \sum_{i=1}^n \delta_{x_i})$ ,  $x_1, \dots, x_n \in \overline{D}$  where  $F(\mu) = \varphi((g_1, \mu), \dots, (g_k, \mu))$ ,  $g_1, \dots, g_k \in C^1(\overline{D})$ ,  $\varphi \in C_b^1(\mathbb{R}^k)$ ,  $k \in \mathbb{Z}_+$ . Also, we take into consideration that every jump of  $X^n$  is  $P_\mu$ -a.e. caused by only one of the  $d$ -dimensional processes  $X_1^n, \dots, X_n^n$ , cf. condition (i) of this theorem. Denoting by  $\text{diam}(D)$  the diameter of  $D$  we get

$$E_\mu (\gamma(\delta)) \leq 3\|f\|^2 \cdot E_\mu |\rho_{-|\alpha|}(X) - \mathbb{1}| + \frac{\text{diam}(D)}{n} 4\|f\| \cdot \|\nabla \varphi\| \cdot \sum_{j=1}^k \|\nabla g_j\|. \quad (5.2)$$

*Step 2* We re-add the index  $n$  to the notation and note that the density function  $m$  depends on the number of particles  $n$ . In this step we aim to prove

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} E_{\mu_n}^n (\gamma_n(\delta)) = 0. \quad (5.3)$$

We note that, independent of  $n \in \mathbb{N}$ , we have  $\|f\| = \|F\|$ . From (5.1) and (5.2) we obtain

$$\begin{aligned}
& \frac{1}{3\|F\|^2} \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} E_{\mu_n}^n (\gamma_n(\delta)) \leq \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} E_{\mu_n}^n \left( \sup_{t \in [0, \delta]} \left| \frac{m_n(X_{-t}^n - Y_{-t}^n \circ u^{-1})}{m_n(u_0^{-1})} \times \right. \right. \\
& \quad \left. \left. \times \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} Z_{-t}^n \circ u^{-1} \right|_i - \mathbb{1} \right| \right)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} E_{\nu_n}^{(n)} \left( \sup_{t \in [0, \delta]} \left| \frac{m_n(X_{-t}^n - Y_{-t}^n)}{m_n(W_0)} \cdot \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} Z_{-t}^n \Big|_i - \mathbb{1} \right| \right| \right) \\
&\leq \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} E_{\nu_n}^{(n)} \left( \sup_{t \in [0, \delta]} \left| \frac{m_n(X_{-t}^n - Y_{-t}^n) - m_n(W_0)}{m_n(W_0)} \right| \right) \\
&\quad + \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} E_{\nu_n}^{(n)} \left( \sup_{t \in [0, \delta]} \left| \frac{m_n(X_{-t}^n - Y_{-t}^n) - m_n(W_0)}{m_n(W_0)} \times \right. \right. \\
&\quad \quad \left. \left. \times \left( \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} Z_{-t}^n \Big|_i - \mathbb{1} \right| \right) \right| \right) \\
&\quad + \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} E_{\nu_n}^{(n)} \left( \sup_{t \in [0, \delta]} \left| \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} Z_{-t}^n \Big|_i - \mathbb{1} \right| \right) \\
&= \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} T_1'(\delta) + \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} T_2'(\delta) + \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} T_3'(\delta). \tag{5.4}
\end{aligned}$$

Let us take a look at the items of the right-hand side. It holds that

$$T_1'(\delta) = \int_{D^n} E_x^{(n)} \left( \sup_{t \in [0, \delta]} \left| m_n(X_{-t}^n - Y_{-t}^n) - m_n(x) \right| \right) dx, \tag{5.5}$$

$$\begin{aligned}
T_2'(\delta) &= \int_{D^n} E_x^{(n)} \left( \sup_{t \in [0, \delta]} \left| (m_n(X_{-t}^n - Y_{-t}^n) - m_n(x)) \times \right. \right. \\
&\quad \left. \left. \times \left( \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} Z_{-t}^n \Big|_i - \mathbb{1} \right| \right) \right| \right) dx \\
&\leq \int_{D^n} E_x^{(n)} \left( \sup_{t \in [0, \delta]} \left| m_n(X_{-t}^n - Y_{-t}^n) - m_n(x) \right| \right) dx \times \\
&\quad \times \operatorname{ess\,sup}_{\Omega} \left( \sup_{t \in [0, \delta]} \left| \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} Z_{-t}^n \Big|_i - \mathbb{1} \right| \right), \tag{5.6}
\end{aligned}$$

and

$$T_3'(\delta) \leq \operatorname{ess\,sup}_{\Omega} \left( \sup_{t \in [0, \delta]} \left| \prod_{i=1}^{n \cdot d} \left| \mathbf{e} + \nabla_{d, W_0} Z_{-t}^n \Big|_i - \mathbb{1} \right| \right). \tag{5.7}$$

We apply the hypotheses (ii) and (iii) of the present proposition to (5.5)-(5.7). From (5.4) we obtain (5.3) in this way.

*Step 3* We finish the proof of part (a). Let  $-\infty < S < T < \infty$  and  $0 < \delta \leq 1$  and let  $(\mathcal{F}_t^n)_{t \in \mathbb{R}}$  denote the filtration generated by  $X^n$  on  $(-\infty, t]$ . For  $0 \leq u \leq \delta$  and  $S \leq t \leq T$  we have

$$E_{\mu_n}^n \left( (f(X_{t+u}^n) - f(X_t^n))^2 \Big| \mathcal{F}_t^n \right) \leq E_{\mu_n}^n (\gamma_n(\delta) \mid \mathcal{F}_t^n).$$

We get now the claim from (5.3), Chapter 3 of [5], Theorem 8.6 and Remark 8.7.

*Step 4* Part (b) follows now from (a) and Chapter 3 of [5], Theorem 9.1.  $\square$

The above theorem and the corollary cover particle systems with a certain abstract drift, Brownian noise such that the drift may not be adapted, and, for given Brownian trajectory, non-random jumps. As discussed in the introduction, the jump mechanism is adopted from particle systems approximating Boltzmann type equations.

The conditions formulated here in order to verify weak compactness of the particle system are of abstract character. They may give a guideline in more concrete situations. Conditions (i) and (ii) refer to the construction of the particle system. They have to be verified by taking into consideration the particular situation. The interesting condition of Theorem 5.1 is (iii). We are interested in the following simple but fairly instructive example.

Let  $\nabla_d$  and  $\Delta_d$  denote the  $d$ -dimensional gradient and Laplace operator. For the sake of simplicity, let us suppose that  $D$  is of Lebesgue measure one which also means that  $\lambda^{n \cdot d}(D^n) \equiv \lambda_F(D^n) = 1$ . Choose  $c \in (0, 1)$  and let  $\hat{H} \equiv \hat{H}_c$  be the set of all probability measures  $d(x) dx$  on  $(\overline{D}, \mathcal{B}(\overline{D}))$  such that

$$d \in C^2(\overline{D}) \text{ with } c \leq d \leq c^{-1} \text{ and } \|\nabla_d d\| \leq c^{-1}. \quad (5.8)$$

Let  $\nu$  be a probability measure on  $(\mathcal{M}_1(\overline{D}), \mathcal{B}(\mathcal{M}_1(\overline{D})))$  such that  $\nu(\mathcal{M}_1(\overline{D}) \setminus \hat{H}) = 0$ . We define the measures  $\tilde{\nu}_n(dx) = m_n(x) dx$  on  $(\overline{D}^n, \mathcal{B}(\overline{D}^n))$  and  $\nu_n(d\mu^n)$  on the set of all empirical measure  $\mu^n = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$ ,  $x_1, \dots, x_n \in \overline{D}$ , by  $\nu_n(d\mu^n) = \tilde{\nu}_n(dx)$ ,  $x = (x_1, \dots, x_n)$  and

$$\begin{aligned} m_n(x) &:= \int_{\mu=d(x) dx \in \hat{H}} \frac{1}{\|d^{\frac{1}{n}}\|_{L^1(D)}^n} \prod_{i=1}^n (d(x_i))^{\frac{1}{n}} \nu(d\mu) \\ &= \int_{\mu=d(x) dx \in \hat{H}} \frac{1}{\|d^{\frac{1}{n}}\|_{L^1(D)}^n} \exp\{(\log(d), \mu^n)\} \nu(d\mu), \quad n \in \mathbb{N}. \end{aligned} \quad (5.9)$$

The following two remarks shall discuss the choice of the initial distribution relative to  $n \in \mathbb{N}$ . On the one hand the distribution of a typical individual particle has order  $d^{1/n}$ . On the other hand, Remark (1) below shows that the overall initial distribution behaves moderately with respect to  $n \in \mathbb{N}$ . In addition, Remark (2) below says that even the gradient of the overall initial density as well as the generator of the Brownian motion part, applied to the overall initial density, behave moderately with respect to  $n \in \mathbb{N}$ . These are important technical features.

**Remarks. (1)** Since  $1 + (\log(d))/n \leq d^{\frac{1}{n}} \leq 1 + (\log(d))/n + ((\log(d))/n)^2$  for sufficiently large  $n \in \mathbb{N}$  for all  $d \in \hat{H}$  by (5.8), we have for such  $n$

$$\left(1 + \frac{1}{n} \int \log(d) dx\right)^n \leq \|d^{\frac{1}{n}}\|_{L^1(D)}^n \leq \left(1 + \frac{1}{n} \int \log(d) dx + \frac{1}{n^2} \int (\log(d))^2 dx\right)^n. \quad (5.10)$$

Relation (5.9) and the weak law of large numbers show now that  $\nu_n$  converges to  $\nu$  as  $n \rightarrow \infty$  in the weak topology of probability measures over  $\mathcal{M}_1(\overline{D})$ .

**(2)** Denoting by  $\nabla$  and  $\Delta$  the  $n \cdot d$ -dimensional gradient and Laplace operator we obtain

$$\nabla m_n(x) = \left( \frac{1}{n} \int_{\mu=d(x) dx \in \hat{H}} \frac{1}{\|d^{\frac{1}{n}}\|_{L^1(D)}^n} \cdot \frac{\nabla_d d(x_i)}{d(x_i)} \prod_{j=1}^n (d(x_j))^{\frac{1}{n}} \nu(d\mu) \right)_{i=1, \dots, n}$$

and

$$\begin{aligned} \frac{1}{2}\Delta m_n(x) = & \frac{1}{n} \left( - \sum_{i=1}^n \int_{\mu=d(x) \, dx \in \hat{H}} \frac{n-1}{\|d^{\frac{1}{n}}\|_{L^1(D)}^n} \cdot \frac{(\nabla_d d(x_i), \nabla_d d(x_i))_{\mathbb{R}^d}}{2n(d(x_i))^2} \prod_{j=1}^n (d(x_j))^{\frac{1}{n}} \boldsymbol{\nu}(d\mu) \right. \\ & \left. + \sum_{i=1}^n \int_{\mu=d(x) \, dx \in \hat{H}} \frac{1}{\|d^{\frac{1}{n}}\|_{L^1(D)}^n} \cdot \frac{\frac{1}{2}\Delta_d d(x_i)}{d(x_i)} \prod_{j=1}^n (d(x_j))^{\frac{1}{n}} \boldsymbol{\nu}(d\mu) \right), \quad n \in \mathbb{N}. \end{aligned}$$

Keeping in mind that the left-hand side of (5.10) is increasing in  $n$  for sufficiently large  $n \in \mathbb{N}$ , we verify the existence of  $c_0 > 0$  and  $n_0 > 0$  such that  $\|d^{\frac{1}{n}}\|_{L^1(D)}^n \geq c_0$  for  $n \geq n_0$  and all  $d \in \hat{H}$  by (5.8). By assumption (5.8) it holds also that

$$\|\nabla m_n\| \leq \frac{n^{-\frac{1}{2}}}{c_0 c^3}, \quad n \geq n_0, \quad (5.11)$$

where  $c$  is the constant in (5.8). It follows that

$$\lim_{n \rightarrow \infty} \int_{D^n} (\nabla m_n, \nabla m_n)_{\mathbb{R}^{n \cdot d}} \, dx = 0 \quad (5.12)$$

By (5.9), (5.10), and the weak law of large numbers we have  $\lim_{n \rightarrow \infty} \int_{D^n} \frac{1}{2} \Delta m_n \, dx = \int_{\mu=d(x) \, dx \in \hat{H}} \int_D \frac{1}{2} \Delta_d \log(d) \, dx \boldsymbol{\nu}(d\mu)$  as well as

$$\limsup_{n \rightarrow \infty} \int_{D^n} \left| \frac{1}{2} \Delta m_n \right| \, dx \leq \int_{\mu=d(x) \, dx \in \hat{H}} \int_D \left| \frac{1}{2} \Delta_d \log(d) \right| \, dx \boldsymbol{\nu}(d\mu) < \infty. \quad (5.13)$$

In order to verify (iii) of Theorem 5.1 under weak additional conditions for the example (5.9) it is beneficial to apply the following proposition.

**Proposition 5.2** *For every  $n \in \mathbb{N}$ , let  $X^n$  be a stochastic process satisfying the hypotheses of Theorem 1.11 or, respectively, of Theorem 1.12. Let  $m_n$  be as in Corollary 1.13. In addition, suppose that for every  $n \in \mathbb{N}$  and  $Q_{\boldsymbol{\nu}_n}^{(n)}$ -a.e.  $W \in \Omega^{(n)}$ ,  $(A^1)^n(W)$  is equi-continuous on  $\mathbb{R}$ . Let  $m_n$ ,  $n \in \mathbb{N}$ , be defined by (5.9). If*

(iv) *for every  $n \in \mathbb{N}$ , there is a random variable  $b_n$  with*

$$|A_{-t}^n - Y_{-t}^n| \leq n^{\frac{1}{2}} \cdot b_n t, \quad t \in [0, 1],$$

*and  $\int_{D^n} E_x^{(n)}(b_n) \, dx$  is bounded in  $n \in \mathbb{N}$*

*then we have (iii) of Theorem 5.1.*

Proof. In condition (iii) of Theorem 5.1 we look at

$$E_{\mathbf{1}_{D^n} \lambda_F}^{(n)} \left( \sup_{t \in [0, \delta]} \left| m_n(X_{-t}^n - Y_{-t}^n) - m_n(x) \right| \right).$$

Let  $\delta \in [0, 1]$ . We have

$$\begin{aligned}
& \sup_{t \in [0, \delta]} \left| m_n(X_{-t}^n - Y_{-t}^n) - m_n(x) \right| \\
& \leq \sup_{t \in [0, \delta]} \left| m_n(W_{-t}^n) - m_n(x) \right| + \sup_{t \in [0, \delta]} \left| m_n(W_{-t}^n + (A_{-t}^n - Y_{-t}^n)) - m_n(W_{-t}^n) \right| \\
& = \sup_{t \in [0, \delta]} T_1''(t) + \sup_{t \in [0, \delta]} T_2''(t). \tag{5.14}
\end{aligned}$$

We extend  $m_n$  outside of  $D^n$  with zero and denote this extension to  $\mathbb{R}^{n \cdot d} \equiv F$  also with  $m_n$ . The random subset  $T_n$  of  $[0, 1]$  in which  $W_{-t} \in D^n$  is  $Q_{\nu_n}$ -a.e. an open subset of  $[0, 1]$  in the trace topology of  $\mathbb{R}$ . Thus  $T_n \cap (0, 1)$  is a countable union of open subintervals of  $(0, 1)$ . According to Itô's formula it holds that

$$\begin{aligned}
T_1''(t) &= \left| m_n(W_{-t}^n) - m_n(x) \right| \\
&= \left| \int_{s \in [0, t] \cap T_n} \langle \nabla m_n(W_{-s}^n), dW_{-s} \rangle_F + \int_{s \in [0, t] \cap T_n} \frac{1}{2} \Delta m_n(W_{-s}^n) ds \right|, \quad t \in [0, \delta].
\end{aligned}$$

By Doob's maximal inequality

$$\begin{aligned}
E_{\mathbb{I}_{D^n} \lambda_F}^{(n)} \left( \sup_{t \in [0, \delta]} |T_1''| \right) &\leq \left( E_{\mathbb{I}_{D^n} \lambda_F}^{(n)} \left( \int_{s \in [0, \delta] \cap T_n} \langle \nabla m_n(W_{-s}^n), dW_{-s} \rangle_F \right)^2 \right)^{1/2} \\
&\quad + E_{\mathbb{I}_{D^n} \lambda_F}^{(n)} \int_{s \in [0, \delta] \cap T_n} \left| \frac{1}{2} \Delta m_n(W_{-s}^n) \right| ds \\
&\leq \left( \int_{s \in [0, \delta]} E_{\mathbb{I}_{D^n} \lambda_F}^{(n)} \langle \nabla m_n(W_{-s}^n), \nabla m_n(W_{-s}^n) \rangle_F ds \right)^{1/2} \\
&\quad + \int_{s \in [0, \delta]} E_{\mathbb{I}_{D^n} \lambda_F}^{(n)} \left| \frac{1}{2} \Delta m_n(W_{-s}^n) \right| ds
\end{aligned}$$

where we treat  $\nabla m_n(W_{-s}^n)$  as well as  $-\frac{1}{2} \Delta m_n(W_{-s}^n)$  as random elements which are zero outside the random set  $T_n$ . This is equivalent to considering  $\nabla m_n$  as well as  $-\frac{1}{2} \Delta m_n$  as functions which are zero outside of  $D^n$ . In this sense we obtain

$$\begin{aligned}
E_{\mathbb{I}_{D^n} \lambda_F}^{(n)} \left( \sup_{t \in [0, \delta]} |T_1''| \right) &\leq e^{1/2} \left( \int_{s \in [0, \delta]} e^{-s/\delta} E_{\mathbb{I}_{D^n} \lambda_F}^{(n)} \langle \nabla m_n(W_{-s}^n), \nabla m_n(W_{-s}^n) \rangle_F ds \right)^{1/2} \\
&\quad + e \int_{s \in [0, \delta]} e^{-s/\delta} E_{\mathbb{I}_{D^n} \lambda_F}^{(n)} \left| \frac{1}{2} \Delta m_n(W_{-s}^n) \right| ds \\
&\leq e^{1/2} \left( \int_{s \in \mathbb{R}_+} e^{-s/\delta} E_{\mathbb{I}_{D^n} \lambda_F}^{(n)} \langle \nabla m_n(W_{-s}^n), \nabla m_n(W_{-s}^n) \rangle_F ds \right)^{1/2} \\
&\quad + e \int_{s \in \mathbb{R}_+} e^{-s/\delta} E_{\mathbb{I}_{D^n} \lambda_F}^{(n)} \left| \frac{1}{2} \Delta m_n(W_{-s}^n) \right| ds.
\end{aligned}$$

Denoting by  $G_\beta^{(n)}$ ,  $\beta > 0$ , the resolvent of the  $n \cdot d$ -dimensional Brownian motion on  $\mathbb{R}^{n \cdot d} = F$  this yields

$$E_{\mathbb{I}_{D^n} \lambda_F}^{(n)} \left( \sup_{t \in [0, \delta]} |T_1''| \right) \leq e^{1/2} \left( \int_{x \in D^n} G_{1/\delta}^{(n)} (\langle \nabla m_n, \nabla m_n \rangle_F)(x) dx \right)^{1/2}$$

$$\begin{aligned}
& +e \int_{x \in D^n} G_{1/\delta}^{(n)} \left| \frac{1}{2} \Delta m_n(x) \right| dx \\
& \leq (e\delta)^{1/2} \left( \int_{D^n} \langle \nabla m_n, \nabla m_n \rangle_F dx \right)^{1/2} + e\delta \int_{D^n} \left| \frac{1}{2} \Delta m_n \right| dx
\end{aligned}$$

where, for the last line, we have used the symmetry of  $G_{1/\delta}^{(n)}$  in  $L^2(F, \lambda_F)$  restricted to the bounded functions of compact support. With (5.12) and (5.13) we get

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} E_{\mathbb{1}_{D^n} \lambda_F}^{(n)} \left( \sup_{t \in [0, \delta]} |T_1''| \right) = 0. \quad (5.15)$$

For  $T_2''$  introduced in (5.14) we have

$$T_2''(t) = \left| m_n(W_{-t}^n + (A_{-t}^n - Y_{-t}^n)) - m_n(W_{-t}^n) \right| \leq \|\nabla m_n\| \cdot |A_{-t}^n - Y_{-t}^n|.$$

Relation (5.11) and condition (iv) of this proposition give directly

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} E_{\mathbb{1}_{D^n} \lambda_F}^{(n)} \left( \sup_{t \in [0, \delta]} |T_2''| \right) = 0. \quad (5.16)$$

The claim follows now from (5.14)-(5.16).  $\square$

## 6 Appendix: Basic Malliavin Calculus for Brownian Motion with Random Initial Data

In this section, we are going to identify the logarithmic derivative relative to a Brownian motion  $B$  with random initial data. We follow the usual concept which means we establish a related Cameron-Martin type formula, a gradient operator, and a stochastic integral as its dual.

It turns out that the whole analysis could be separated into the well known Wiener space case, which is the case with initial datum zero, and an independent part related to the finite dimensional initial variable. This is due to the fact that we assume  $B$  at time zero to be independent of the process at all other times. We treat both parts simultaneously and mention that the pure Wiener space part is presented here in a very compressed way. For this case it is recommended to compare the following with [19] and the thorough introduction to differentiation and integration on the Wiener space in [20], Appendix B. For the case including the random initial condition, we refer to [2] as a source of stochastic calculus relative to abstract non-Gaussian measure spaces. We assume the process  $B = ((B_s)_{s \in \mathcal{T}}, (\mathcal{F}_u^v \times \sigma(B_0))_{u, v \in \mathcal{T}, u \leq v}, (P_x)_{x \in F}), \mathcal{T}$  either  $[0, 1]$  or  $[-1, 1]$ , to be endowed with an initial random variable  $B_0$  independent of  $(B_s - B_0)_{s \in \mathcal{T} \setminus \{0\}}$  whose distribution is denoted by  $\nu$ . Let us assume that  $\nu$  admits a density

$$0 < m \in C^1(F) \quad \text{with} \quad \lim_{\|x\|_F \rightarrow \infty} m(x) = 0. \quad (6.1)$$

Also, let us introduce the shift by  $y \in F$ ,

$$U_y(x) = x + y, \quad x \in F.$$

Furthermore, let us assume that there exist  $q \in (1, \infty)$  such that, for all  $y \in F$ ,

$$\frac{m \circ U_y}{m} \in L^q(F, \boldsymbol{\nu}) \quad \text{and} \quad \frac{d}{d\lambda} \Big|_{\lambda=0} \frac{m \circ U_{\lambda y}}{m} = \left\langle \frac{\nabla m}{m}, y \right\rangle_F \quad \text{exists in } L^q(F, \boldsymbol{\nu}). \quad (6.2)$$

Let  $\Omega := C(\mathcal{T}; F)$  denote the set of trajectories associated with  $(B_s)_{s \in \mathcal{T}}$  starting from  $s = 0$  to either direction in case of  $\mathcal{T} = [-1, 1]$ . Also introduce  $P_\nu := \int P_x \boldsymbol{\nu}(dx)$  and let  $E_\nu$  denote the corresponding mathematical expectation. Furthermore, we suppose that the filtration  $\{\mathcal{F}_u^v = \sigma(W_\alpha - W_\beta : u \leq \alpha, \beta \leq v) \times \sigma(W_0) : u, v \in \mathcal{T}, u < v\}$  is completed by the  $P_\nu$ -completion of the  $\sigma$ -algebra  $\mathcal{F}$  of all Borel subsets of  $\Omega$  relative to uniform convergence on  $\mathcal{T}$ .

**Cameron-Martin type space, embedding, shift relative to  $B_0$ .** Let

$$H := \{(f, x) : f \in L^2(\mathcal{T}; F), x \in F\},$$

be equipped with the inner product

$$\langle (f, x), (g, y) \rangle_H := \langle f, g \rangle_{L^2} + \langle x, y \rangle_F.$$

Furthermore, define

$$W := \{(\omega, x) : \omega \in C(\mathcal{T}; F), \omega(0) = 0, x \in F\}.$$

We embed  $H$  into  $W \equiv C(\mathcal{T}; F)$  by

$$j(f, x) := \left( \int_0^\cdot f(s) ds, x \right) \equiv x + \int_0^\cdot f(s) ds, \quad (f, x) \in H.$$

**Cameron-Martin type formula.** Let  $\mathcal{M}^{f,s}(\mathcal{T}; F)$  denote the set of all  $F$ -valued finite signed measures on  $(\mathcal{T}, \mathcal{B}(\mathcal{T}))$  and

$$W^* := \{(\omega^*, x^*) : \omega^* \in \mathcal{M}^{f,s}(\mathcal{T}; F) \text{ with } \omega^*(\mathcal{T}) = 0, x^* \in F\}.$$

We have

$$\begin{aligned} {}_{W^*} \langle (\omega^*, x^*), (\omega, x) \rangle_W &= \int_{\mathcal{T}} \langle \omega, d\omega^* \rangle_F + \langle x, x^* \rangle_F \\ &= \int_{\mathcal{T}} \langle \omega^*((s, 1]), d\omega_s \rangle_F + \langle x, x^* \rangle_F. \end{aligned}$$

For  $(\omega, x) \in W$  and  $(f, y) \in H$  introduce

$$S_{(f,y)}(\omega, x) := \left( \omega + \int_0^\cdot f(s) ds, U_y(x) \right)$$

and

$$\mathcal{E}_{(f,y)}(\omega, x) := \exp \left\{ \int_{\mathcal{T}} \langle f(s), d\omega_s \rangle_F - \frac{1}{2} \int_{\mathcal{T}} \langle f(s), f(s) \rangle_F ds \right\} \frac{m(U_{-y}(x))}{m(x)}.$$

**Theorem 6.1** *We have*

$$(i) \quad \frac{dP_\nu \circ S_{(f,y)}}{dP_\nu}((\omega, x)) = \mathcal{E}_{(-f,-y)}(\omega, x).$$

Proof. Let  $(f, y) \in H$ . First, verify that the map

$$L^1(W, P_\nu) \ni \Phi \longrightarrow E_\nu \left( \exp \{ i_{W^*} \langle \cdot, S_{(f,y)} \rangle_W \} \Phi \right)$$

is an injection. Then focus on

$$\begin{aligned} & E_\nu \left( \exp \{ i_{W^*} \langle (\omega^*, x^*), S_{(f,y)}(\omega, x) \rangle_W \} \mathcal{E}_{(-f,-y)}(\omega, x) \right) \\ &= E_\nu \left( \exp \left\{ i \left( \int_{\mathcal{T}} \left\langle \omega^*((s, 1]), d \left( \omega_s + \int_0^s f(v) dv \right) \right\rangle_F + \langle x^*, U_y(x) \rangle_F \right) \right. \right. \\ &\quad \left. \left. - \int_{\mathcal{T}} \langle f(s), d\omega_s \rangle_F - \frac{1}{2} \int_{\mathcal{T}} \langle f(s), f(s) \rangle_F ds \right\} \frac{m(U_y(x))}{m(x)} \right) \\ &= \exp \left\{ -\frac{1}{2} \int_{\mathcal{T}} \langle \omega^*((s, 1]), \omega^*((s, 1]) \rangle_F ds \right\} \times \\ &\quad \times E_\nu \left( \exp \{ i \langle x^*, U_y(x) \rangle_F \} \frac{m(U_y(x))}{m(x)} \right) \\ &= \exp \left\{ -\frac{1}{2} \int_{\mathcal{T}} \langle \omega^*((s, 1]), \omega^*((s, 1]) \rangle_F ds \right\} \times \\ &\quad \times \int \exp \{ i \langle x^*, U_y(z) \rangle_F \} m(U_y(z)) dz \\ &= \exp \left\{ -\frac{1}{2} \int_{\mathcal{T}} \langle \omega^*((s, 1]), \omega^*((s, 1]) \rangle_F ds \right\} \int \exp \{ i \langle x^*, z \rangle_F \} m(z) dz. \end{aligned}$$

The claim is an immediate consequence of the independence of the right-hand side of  $(f, y)$ .  
□

**Corollary 6.2** *For all bounded measurable  $\varphi : W \rightarrow \mathbb{R}$ , we have*

$$(ii) \quad E_\nu \left( \varphi((\omega, x) + j(f, U_y(x) - x)) \cdot \mathcal{E}_{(-f,-y)}(\omega, x) \right) = E_\nu \varphi(\omega, x) \text{ and}$$

$$(iii) \quad E_\nu \varphi((\omega, x) + j(f, U_y(x) - x)) = E_\nu \left( \varphi(\omega, x) \cdot \mathcal{E}_{(f,y)}(\omega, x) \right).$$

**Gradient operator.** Let  $k \in \mathbb{N}$ ,  $l_1, \dots, l_k \in W^*$ ,  $f \in C_b^1(\mathbb{R}^k)$ , and

$$\varphi(\omega, x) := f(l_1(\omega, x), \dots, l_k(\omega, x)). \quad (6.3)$$

In other words, let  $\varphi$  be a bounded differentiable cylindrical function on  $W$ . As usual, define

$$\begin{aligned} \mathbb{D}\varphi(\omega, x) &:= \sum_{i=1}^k f_{x_i}(l_1(\omega, x), \dots, l_k(\omega, x)) \cdot_{W^*} \langle l_i, \cdot \rangle_W \\ &\equiv \sum_{i=1}^k f_{x_i}(l_1(\omega, x), \dots, l_k(\omega, x)) \cdot l_i \end{aligned} \quad (6.4)$$

where  $f_{x_i}$  is the partial derivative of  $f$  with respect to the  $i$ -th entry,  $i \in \{1, \dots, k\}$ . Let  $\mathcal{C}$  denote the set of all cylindrical functions of the form (6.3). As a particular case, let us assume the linear functionals to be of the form  $l_i(\omega, x) := \langle a_i, \omega(t_i) \rangle_F + \langle b_i, x \rangle_F$ ,  $a_i, b_i \in F$ ,  $t_1 \leq \dots \leq t_k$ , all in  $\mathcal{T}$ . Then

$$\begin{aligned} \mathbb{D}\varphi(\omega, x)(\rho, y) &= \sum_{i=1}^k f_{x_i} (\langle a_1, \omega(t_1) \rangle_F + \langle b_1, x \rangle_F, \dots, \langle a_k, \omega(t_k) \rangle_F + \langle b_k, x \rangle_F) \cdot (\langle a_i, \rho(t_i) \rangle_F + \langle b_i, y \rangle_F), \\ &(\rho, y) \in W, \end{aligned} \quad (6.5)$$

which makes it comparable with the common representations for the pure Wiener process case as, for example, presented in [12]. In either case we have the usual relation to directional derivatives

$$\mathbb{D}\varphi(\omega, x)(\rho, y) \equiv {}_{W^*}\langle \mathbb{D}\varphi(\omega, x), (\rho, y) \rangle_W = \frac{\partial \varphi(\omega, x)}{\partial (\rho, y)}, \quad (\rho, y) \in jH = \{jh : h \in H\}.$$

**Proposition 6.3** *Let  $q$  be the number used in (6.2) and  $1/p + 1/q \leq 1$ .*

(a) *The set of all cylindrical functions*

$$\varphi(\omega, x) = f(\langle a_1, \omega(t_1) \rangle_F + \langle b_1, x \rangle_F, \dots, \langle a_k, \omega(t_k) \rangle_F + \langle b_k, x \rangle_F)$$

*$f \in C_b^1(\mathbb{R}^k)$ ,  $a_i, b_i \in F$ ,  $i \in \{1, \dots, k\}$ ,  $t_1 \leq \dots \leq t_k$ , all in  $\mathcal{T}$ , is dense in  $L^p(W, P_\nu)$ .*

(b) *Let  $\tilde{\mathcal{C}} \subseteq \mathcal{C}$  be a set of cylindrical functions which is dense in  $L^p(W, P_\nu)$ . Then the operator*

$$(D, \tilde{\mathcal{C}}) := (j^* \circ \mathbb{D}, \tilde{\mathcal{C}}) \quad (6.6)$$

*is closable on  $L^p(W, P_\nu; H)$ .*

Proof. (a) This is explained in [19] Preliminaries 5 i), for the pure Wiener space case. Anything else is trivial.

(b) Assume  $\tilde{\mathcal{C}} \ni \varphi_n \xrightarrow{n \rightarrow \infty} 0$  in  $L^p(W, P_\nu)$  and  $D\varphi_n$ ,  $n \in \mathbb{N}$ , be Cauchy in  $L^p(W, P_\nu; H)$ . Then  $D\varphi_n \xrightarrow{n \rightarrow \infty} \xi$  in  $L^p(W, P_\nu; H)$  for some  $\xi \in L^p(W, P_\nu; H)$ . We have to show  $\xi = 0$   $P_\nu$ -a.e. For  $\varphi \in \tilde{\mathcal{C}}$  and  $(f, y) \in H$ , we have by Corollary 6.2 (iii)

$$\begin{aligned} E_\nu (\langle D\varphi_n, (f, y) \rangle_H \cdot \varphi) &= E_\nu ({}_{W^*}\langle \mathbb{D}\varphi_n, j(f, y) \rangle_W \cdot \varphi) \\ &= \frac{d}{d\lambda} \Big|_{\lambda=0} E_\nu (\varphi_n ((\omega, x) + \lambda j(f, y)) \cdot \varphi(\omega, x)) \\ &= \frac{d}{d\lambda} \Big|_{\lambda=0} E_\nu \left( \varphi_n(\omega, x) \cdot \varphi((\omega, x) - \lambda j(f, y)) \times \right. \\ &\quad \left. \times \exp \left\{ \lambda \int_{\mathcal{T}} f(s) d\omega_s - \frac{\lambda^2}{2} \int_{\mathcal{T}} f^2(s) ds \right\} \cdot \frac{m(U_{-\lambda y}(x))}{m(x)} \right). \end{aligned} \quad (6.7)$$

The derivative of the last line with respect to  $\lambda$  at  $\lambda = 0$  exists in  $L^q(W, P_\nu)$  by (6.2) and is equal to

$$\frac{d}{d\lambda} \Big|_{\lambda=0} \mathcal{E}_{\lambda(f, y)}(\omega, x) = \int_{\mathcal{T}} f d\omega - \left\langle \frac{\nabla m(x)}{m(x)}, y \right\rangle_F. \quad (6.8)$$

Therefore,

$$\begin{aligned} E_\nu (\langle D\varphi_n, (f, y) \rangle_H \cdot \varphi) \\ = E_\nu (\varphi_n \cdot_{W^*} \langle \mathbb{D}\varphi, -j(f, y) \rangle_W) + E_\nu \left( \varphi_n \cdot \varphi \cdot \frac{d}{d\lambda} \Big|_{\lambda=0} \mathcal{E}_{\lambda(f, y)} \right). \end{aligned} \quad (6.9)$$

Letting  $n \rightarrow \infty$  we obtain  $E_\nu (\langle \xi, (f, y) \rangle_H \cdot \varphi) = 0$  for all  $\varphi \in \tilde{\mathcal{C}}$  and  $(f, y) \in H$  from (6.9). Thus  $\xi = 0$   $P_\nu$ -a.e. In other words,  $D$  is closable on  $L^p(W, P_\nu; H)$ .  $\square$

**Definition 6.4** Let  $q$  be the number used in (6.2) and  $1/p + 1/q \leq 1$ . We say  $\varphi \in \text{Dom}_p(D)$  if there is a sequence  $\varphi_n \in \mathcal{C}$ ,  $n \in \mathbb{N}$ , such that  $\mathcal{C} \ni \varphi_n \xrightarrow{n \rightarrow \infty} \varphi$  in  $L^p(W, P_\nu)$  and  $D\varphi_n$ ,  $n \in \mathbb{N}$ , is Cauchy in  $L^p(W, P_\nu; H)$ . In this case

$$D\varphi := \lim_{n \rightarrow \infty} D(\varphi_n).$$

Let  $D_{p,1} \equiv D_{p,1}(P_\nu)$  be the space of all  $\varphi \in \text{Dom}_p(D)$  equipped with the norm  $\|\varphi\|_{p,1} := \|\varphi\|_{L^p(W, P_\nu)} + \|D\varphi\|_{L^p(W, P_\nu; H)}$ .

### Gradient and directional derivative

**Proposition 6.5** (a) Let  $q$  be the number used in (6.2) and  $1/p + 1/q \leq 1$ . Furthermore, let  $\psi \in D_{p,1}$  and  $k \in H$ . Suppose the existence of the limit

$$(i) \quad \frac{\partial \psi}{\partial jk} = \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \left( \psi(\cdot + \lambda jk) - \psi \right)$$

in  $L^p(W, P_\nu)$ . Then

$$\frac{\partial \psi}{\partial jk} = \langle D\psi, k \rangle_H \in L^p(W, P_\nu). \quad (6.10)$$

(b) Let  $1 < p < \infty$  and  $\psi \in L^p(W, P_\nu)$ . Assume the existence of the limit (i) for all  $k \in H$  in the measure  $P_\nu$ . Also suppose the existence of  $C \in L^p(W, P_\nu; H)$  such that

$$(ii) \quad \frac{\partial \psi}{\partial jk} = \langle C, k \rangle_H \quad \text{for all } k \in H \text{ and}$$

$$(iii) \quad \psi((\omega, x) + jk) - \psi((\omega, x)) = \int_0^1 \langle C((\omega, x) + tk), k \rangle_H dt \quad \text{for all } k \in H \text{ and } P_\nu\text{-a.e.} \\ (\omega, x) \in W.$$

Then  $\psi \in D_{p,1}$ ,  $C = D\psi$ , and we have

$$\frac{\partial \psi}{\partial jk} = \langle D\psi, k \rangle_H \in L^p(W, P_\nu). \quad (6.11)$$

Proof. (a) Let  $\varphi \in \mathcal{C}$  and let  $\psi_n \in \mathcal{C}$ ,  $n \in \mathbb{N}$ , be a sequence with  $\psi_n \xrightarrow{n \rightarrow \infty} \psi$  in  $D_{p,1}$ . Following (6.7)-(6.9) back and forth we obtain for  $k = (f, y) \in H$

$$\begin{aligned}
E_\nu(\langle D\psi, k \rangle_H \cdot \varphi) &= \lim_{n \rightarrow \infty} E_\nu(\langle D\psi_n, k \rangle_H \cdot \varphi) \\
&= \lim_{n \rightarrow \infty} \left( -E_\nu(\psi_n \cdot \langle D\varphi, k \rangle_H) + E_\nu\left(\psi_n \cdot \varphi \cdot \frac{d}{d\lambda} \Big|_{\lambda=0} \mathcal{E}_{\lambda(f,y)}\right) \right) \\
&= -E_\nu(\psi \cdot \langle D\varphi, k \rangle_H) + E_\nu\left(\psi \cdot \varphi \cdot \frac{d}{d\lambda} \Big|_{\lambda=0} \mathcal{E}_{\lambda(f,y)}\right) \\
&= \frac{d}{d\lambda} \Big|_{\lambda=0} E_\nu(\psi(\cdot + \lambda jk) \cdot \varphi) \\
&= E_\nu\left(\frac{d}{d\lambda} \Big|_{\lambda=0} \psi(\cdot + \lambda jk) \cdot \varphi\right),
\end{aligned}$$

the last line by hypothesis (i). From here, we conclude (6.10).

(b) Let us first assume that the condition (i) in the measure and the conditions (ii), (iii) are satisfied only for  $C = (C_1, 0)$  with  $C_1 \in L^p(W, P_{\delta_0}; L^2(\mathcal{T}; F))$  and  $k = (f, 0)$  with  $f \in L^2(\mathcal{T}; F)$ . For the conclusion (6.11) for all such  $k = (f, 0)$ , we refer to [20], Appendix B.6. Now part (b) of the proposition follows from the fact that (i) in the measure and (ii), (iii) together with the definition of  $\nu(dx) = m(x)dx$  in (6.1) implies (6.11) for  $k = (0, x)$  with  $x \in F$  and  $C = (0, C_2)$  where  $C_2 \in L^p(W, \nu; F)$ . For this one may also consult [9], Theorem 1.11.  $\square$

**The stochastic integral** We are going to define an anticipating integral which in its specification to the adapted case on the Wiener space is the Itô integral. The following definition takes into consideration the Definition 6.4 in which the spaces  $D_{p,1}$  are meaningfully introduced only for  $1/p \leq 1 - 1/q$ .

**Definition 6.6** Let  $q$  be the number appearing in (6.2) and  $1/p + 1/q = 1$ . Let  $q' \geq p$  and  $1/p' + 1/q' = 1$ . Furthermore, let  $\xi \in L^{p'}(W, P_\nu; H)$ . We say that  $\xi \in \text{Dom}_{p'}(\delta)$  if there exists  $c_{p'}(\xi) > 0$  such that

$$E_\nu(\langle D\varphi, \xi \rangle_H) \leq c_{p'}(\xi) \cdot \|\varphi\|_{L^{q'}(W, P_\nu)}, \quad \varphi \in D_{q',1}.$$

In this case, we define the *stochastic integral*  $\delta(\xi)$  by

$$E_\nu(\delta(\xi) \cdot \varphi) = E_\nu(\langle \xi, D\varphi \rangle_H), \quad \varphi \in D_{q',1}.$$

Replacing  $\nu$  by  $\delta_0$  and concentrating on the case  $\mathcal{T} = [0, 1]$ ,  $\delta(\xi)$  is the *Skorohod integral*.

**Theorem 6.7** Let  $q$  appearing in (6.2) and  $1/p + 1/q = 1$ .

(a) Let  $(f, y) \in H$ . We have  $(f, y) \in \text{Dom}_q(\delta)$  and

$$\delta(f, y) = \int_{\mathcal{T}} \langle f, d\omega \rangle_F - \left\langle y, \frac{\nabla m(x)}{m(x)} \right\rangle_F.$$

(b) Let  $[s_1, s_2] \subseteq \mathcal{T}$ ,  $i, j \in \{1, \dots, n \cdot d\}$ , and  $a \in D_{q',1}$  for some  $q' > p$ . Then  $a(\omega, x) \cdot (e_j \mathbb{I}_{[s_1, s_2]}, e_i) \in \text{Dom}_w(\delta)$  with  $1/w = 1/q' + 1/q$  and

$$\begin{aligned} & \delta \left( a \cdot (e_j \mathbb{I}_{[s_1, s_2]}, e_i) \right) (\omega, x) \\ &= a(\omega, x) \cdot (\omega_{s_2} - \omega_{s_1})_j - a(\omega, x) \cdot \frac{(\nabla m(x))_i}{m(x)} - \langle Da, (e_j \mathbb{I}_{[s_1, s_2]}, e_i) \rangle_H. \end{aligned} \quad (6.12)$$

Assume the existence of the directional derivatives

$$\left. \frac{d}{d\lambda} \right|_{\lambda=0} a \left( \omega + \lambda \int_0^\cdot e_j \mathbb{I}_{[s_1, s_2]}(s) ds, x \right) \quad \text{and} \quad \left. \frac{d}{d\lambda} \right|_{\lambda=0} a(\omega, x + \lambda \cdot e_i).$$

in  $L^{q'}(W, P_\nu)$ . Then

$$\begin{aligned} \delta \left( a \cdot (e_j \mathbb{I}_{[s_1, s_2]}, e_i) \right) (\omega, x) &= a(\omega, x) \cdot (\omega_{s_2} - \omega_{s_1})_j - a(\omega, x) \cdot \frac{(\nabla m(x))_i}{m(x)} \\ &\quad - \left. \frac{d}{d\lambda} \right|_{\lambda=0} a \left( \omega + \lambda \int_0^\cdot e_j \mathbb{I}_{[s_1, s_2]}(s) ds, x \right) - \left. \frac{d}{d\lambda} \right|_{\lambda=0} a(\omega, x + \lambda \cdot e_i). \end{aligned} \quad (6.13)$$

Proof. (a) Looking at (6.9), choosing  $\varphi = 1$  there, and renaming  $\varphi_n$  by  $\psi \in \mathcal{C}$  we get

$$\begin{aligned} E_\nu (\langle D\psi, (f, y) \rangle_H) &= E_\nu \left( \psi \cdot \left. \frac{d}{d\lambda} \right|_{\lambda=0} \mathcal{E}_{\lambda(f, y)} \right) \\ &\leq \|\psi\|_{L^p(W, P_\nu)} \cdot \left\| \left. \frac{d}{d\lambda} \right|_{\lambda=0} \mathcal{E}_{\lambda(f, y)} \right\|_{L^q(W, P_\nu)}. \end{aligned}$$

Part (a) follows now by taking now into consideration  $q \geq 2$  and (6.8),

$$\left. \frac{d}{d\lambda} \right|_{\lambda=0} \mathcal{E}_{\lambda(f, y)}(\omega, x) = \int_{\mathcal{T}} f d\omega - \left\langle \frac{\nabla m(x)}{m(x)}, y \right\rangle_F.$$

(b) Relation (6.12) is a consequence of part (a) and the following. For  $\xi \in H$ ,  $\alpha \in D_{q,1}$  we have  $\alpha\delta(\xi) - \langle D\alpha, \xi \rangle_H \in L^w$  where  $1/w = 1/q + 1/q'$  and

$$\alpha\xi \in \text{Dom}_w(\delta) \quad \text{and} \quad \delta(\alpha\xi) = \alpha\delta(\xi) - \langle D\alpha, \xi \rangle_H.$$

Indeed the latter becomes evident for  $\varphi \in \mathcal{C}$  by adapting a calculation from [19], Section 1.1,

$$\begin{aligned} E_\nu \left( \langle a\xi, D\varphi \rangle_H \right) &= E_\nu \left( \langle \xi, aD\varphi \rangle_H \right) \\ &= E_\nu \left( \langle \xi, D(a\varphi) - \varphi Da \rangle_H \right) \\ &= E_\nu \left( a\delta(\xi) \cdot \varphi \right) - E_\nu \left( \langle Da, \xi \rangle_H \cdot \varphi \right). \end{aligned}$$

Relation (6.13) follows now from Proposition 6.5 (a).  $\square$

Let us continue with an object being, up to its signum, nothing but the stochastic integral.

**Definition 6.8** Let  $h \in \text{Dom}_p(\delta)$  for some  $p \in (1, \infty)$ . We define the *logarithmic derivative*  $\beta_{jh}$  of the measure  $P_\nu$  in direction of  $jh$  by  $\beta_{jh} = -\delta(h)$ .

**The  $L^2$ -integral** There is another type of anticipating integral whose specification to the adapted case on the Wiener space is the Stratonovich integral.

**Definition 6.9** (a) Let  $L_{2,1}$  denote the space of all  $(f, y) \in L^2(W, P_\nu; H)$  such that

(i)  $f_t, y \in D_{2,1}$  for a.e.  $t \in \mathcal{T}$ , and  $P_\nu$ -a.e. it holds that

(ii)  $\int_{t \in \mathcal{T}} \|Df_t\|_H^2 dt + \|Dy\|_H^2 < \infty$ .

(b) Let  $\varphi_i, i \in \mathbb{N}$ , be a complete orthonormal basis of  $L^2(\mathcal{T}; F)$  and  $(f, y) \in L_{2,1}$ . The  $L^2$ -integral

$$\int_{\mathcal{T}} \langle (f, y) \circ d(\omega, x) \rangle_F \equiv \int_{\mathcal{T}} \langle f \circ d\omega \rangle_F - \left\langle y, \frac{\nabla m(x)}{m(x)} \right\rangle_F$$

is defined by

$$\sum_{i=1}^{\infty} \langle f, \varphi_i \rangle_{L^2(\mathcal{T}; F)} \cdot \int_{\mathcal{T}} \langle \varphi_i, d\omega \rangle_F - \left\langle y, \frac{\nabla m(x)}{m(x)} \right\rangle_F$$

provided that this sum converges in the measure  $P_\nu$  independent of the chosen orthonormal basis  $\varphi_i, i \in \mathbb{N}$ , of  $L^2(\mathcal{T}; F)$ .

Replacing  $\nu$  by  $\delta_0$  and choosing  $\mathcal{T} = [0, 1]$ ,  $\int_{\mathcal{T}} \langle f \circ d\omega \rangle_F$  is the *Ogawa integral*.

**Theorem 6.10** We have the fundamental relation between the stochastic integral and the  $L^2$ -integral. Let  $(f, y) \in L_{2,1}$  and suppose that  $P_\nu$ -a.e.  $D(f, y)(\cdot) : H \rightarrow H$  is a nuclear operator. Then

$$\int_{\mathcal{T}} (f, y) \circ d(\omega, x) = \delta(f, y) + \text{trace}(D(f, y)).$$

Proof. For  $\nu$  replaced by  $\delta_0$  and  $\mathcal{T} = [0, 1]$ , this is due to [12], Section 3.1.2. Anything else is left as an exercise.  $\square$

For expositions beyond the scope of this paper, we refer to [1], [2], [3], [10], [12], [13], [16], [17], [18].

## References

- [1] V. BOGACHEV, *Gaussian measures*, Mathematical Surveys and Monographs, Vol. **62**, Providence: AMS 1998.
- [2] V. BOGACHEV, E. MAYER-WOLF, Absolutely continuous flows generated by Sobolev class vector fields in finite and infinite dimensions. *J. Funct. Anal.* **167** (1999) 1-68.
- [3] R. BUCKDAHN, Anticipative Girsanov transformations and Skorohod stochastic differential equations. *Mem. Amer. Math. Soc.* **111** No. 553 (1994).

- [4] S. CAPRINO, M. PULVIRENTI, W. WAGNER, Stationary particle systems approximating stationary solutions to the Boltzmann equation. *SIAM J. Math. Analysis* **29** No. 4 (1998), 913–934.
- [5] S. N. ETHIER, T. KURTZ, *Markov processes, Characterization and convergence*, New York Chichester Brisbane Toronto Singapore: John Wiley 1986.
- [6] I. I. GIHMAN, A. V. SKOROHOD, On densities of probability measures in function spaces. *Russ. Math. Surv.* **21** No. 6 (1966), 83-156.
- [7] C. GRAHAM, S. MÉLÉARD, Stochastic particle approximations for generalized Boltzmann models and convergence estimates. *Ann. Probab.* **25** No. 1 (1997), 115-132.
- [8] P. IMKELLER, On the law of orthogonal projectors on eigenspaces of Lyapunov exponents of linear stochastic differential equations. In: H.-J. Engelbert, H. Föllmer, J. Zabczyk, *Stochastic Processes and Related Topics, Stochastics Monographs* **10**, 33-47, Amsterdam: Gordon and Breach Publishers 1996.
- [9] A. KUFNER, B. OPIC, How to define reasonably weighted Sobolev spaces. *Comment. Math. Univ. Carol.* **25** No. 3 (1984), 537-554.
- [10] A. M. KULIK, A. YU. PILIPENKO, Nonlinear transforms of smooth measures on infinite-dimensional spaces. *Ukrainian Math. J.* **52** No. 9 (2000), 1403-1431.
- [11] E. MAYER-WOLF, M. ZAKAI, The divergence of Banach space valued random variables on Wiener space. *Probab. Theory Related Fields* **132** No. 2 (2005), 291-320.
- [12] D. NUALART, *The Malliavin Calculus and Related Topics*, 2nd edition, Berlin Heidelberg: Springer 2006.
- [13] D. NUALART, M. ZAKAI, On the relation between the Stratonovich and Ogawa integrals. *Ann. Probab.* **17** No. 4 (1989), 1536–1540.
- [14] YU. V. PROKHOROV, A. N. SHIRYAEV (Eds.), *Probability Theory III: Stochastic Calculus*, Encyclopaedia of Mathematical Sciences, Berlin Heidelberg: Springer 1998.
- [15] P. E. PROTTER, *Stochastic integration and differential equations*, 2nd edition, version 2.1, Berlin Heidelberg: Springer 2005.
- [16] J. ROSINSKI, On stochastic integration by series of Wiener integrals. *Appl. Math. Optim.* **19** (1989), 137-155.
- [17] I. SHIGEYAMA, *Stochastic analysis*, Translations of Mathematical Monographs, Providence: AMS 2000.
- [18] O. G. SMOLYANOV, H. V. WEIZSÄCKER, Smooth probability measures and associated differential operators. *Inf. Dimens. Anal. Quantum Probab. Relat. Top.* **2** No. 1 (1999), 51-78.
- [19] A. S. ÜSTÜNEL, An introduction to analysis on Wiener space, *Lecture Notes in Math.* 1610, Berlin: Springer 1995.
- [20] A. S. ÜSTÜNEL, M. ZAKAI, *Transformation of Measure on Wiener Space*, Berlin Heidelberg: Springer 2000.